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Automatic Control Systems - Part III: Root Locus Technique

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Automatic Control Systems

Part III:

Root Locus Technique

By Shih-Min Hsu, Ph.D., P.E.

VI. Root Locus Design Method

Since the performance of a closed-loop feedback system can be adjusted by changing one or more parameters, the locations of the roots of the characteristic equation should be evaluated. In the design and analysis of control systems, control engineers often need to investigate a system's performance when one or more of its parameters vary over a given range. Referring to the transfer function of a typical negative feedback closed-loop system as discussed previously, namely,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \quad (6-1)$$

The characteristic equation of the closed-loop system is obtained by setting the denominator polynomial of the transfer function, $\frac{C(s)}{R(s)}$, to zero. Therefore, the roots of the characteristic equation must satisfy

$$1 + G(s)H(s) = 0 \quad (6-2)$$

Let the open-loop transfer function $G(s)H(s)$ express as

$$G(s)H(s) = \frac{KN(s)}{D(s)} = \frac{K[s^m + b_1s^{m-1} + \dots + b_{m-1}s + b_m]}{s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n} \quad (6-3)$$

where $N(s)$ and $D(s)$ are finite polynomials and K is the open-loop gain. Moreover, m is the degree of $N(s)$ and n is the degree of $D(s)$. Equation (6-2) can be replaced as follows

$$1 + \frac{KN(s)}{D(s)} = 0 \quad (6-4)$$

Furthermore, Equation (6-4) can be represented with

$$\frac{N(s)}{D(s)} = -\frac{1}{K} \quad (6-5)$$

The root locus is the locus of values of s for which Equation (6-5) holds for some positive values of K . Namely, if $K > 0$,

$$\left| \frac{N(s)}{D(s)} \right| = \frac{1}{K} \quad (6-6)$$

and

$$\angle \frac{N(s)}{D(s)} = (2k + 1)\pi \quad (\text{or odd multiple of } 180^\circ) \quad (6-7)$$

where $k=0, \pm 1, \pm 2, \pm 3, \dots$.

It is worth mentioning that the portion of the root locus when K varies from $-\infty$ to 0 ($K < 0$) is called Complementary Root Locus (CRL). For $K > 0$, the portion of Root

Locus is called Direct Root Locus, or simply, Root Locus (RL). This course will only discuss the part that $K > 0$.

If K does not appear as a multiple factor of $G(s)H(s)$ as shown in (6-3), one may need to rearrange the function into the following form:

$$F(s) = D(s) + KN(s) = 0 \tag{6-8}$$

Example 6-1: Consider the open-loop transfer function of a control system is

$$G(s)H(s) = \frac{s^2 + (3 + 2K)s + 5}{s(s+1)(s+2)}$$

Find $N(s)$ and $D(s)$ in terms of $D(s) + KN(s) = 0$.

Solution:

The characteristic equation of the closed-loop system is

$$1 + G(s)H(s) = 1 + \frac{s^2 + (3 + 2K)s + 5}{s(s+1)(s+2)} = 0$$

Then,

$$s(s+1)(s+2) + s^2 + (3 + 2K)s + 5 = s(s+1)(s+2) + s^2 + 3s + 5 + 2Ks = 0$$

Divide both sides by the terms that do not contain K , namely, $s(s+1)(s+2) + s^2 + 3s + 5$

$$1 + \frac{2Ks}{s(s+1)(s+2) + s^2 + 3s + 5} = 0$$

Therefore,

$$N(s) = 2s$$

and

$$D(s) = s(s+1)(s+2) + s^2 + 3s + 5 = s^3 + 4s^2 + 5s + 5 \quad \blacklozenge$$

The open-loop transfer function of a closed-loop system may sometimes express in terms of the zeros and poles. In this case, Equation (6-3) can be expressed as

$$G(s)H(s) = \frac{KN(s)}{D(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{6-9}$$

where the zeros ($-z_1, -z_2, \dots, -z_m$) and poles ($-p_1, -p_2, \dots, -p_n$) of $G(s)H(s)$ are real or in complex-conjugate pairs.

Example 6-2: For the open-loop transfer function of a given system

$$G(s)H(s) = \frac{K(s+2)}{(s+1)(s+3)}$$

Locate the zero(s) and pole(s) on the s-plane.

Solution:

There are one zero and two poles for the open-loop transfer function, namely,

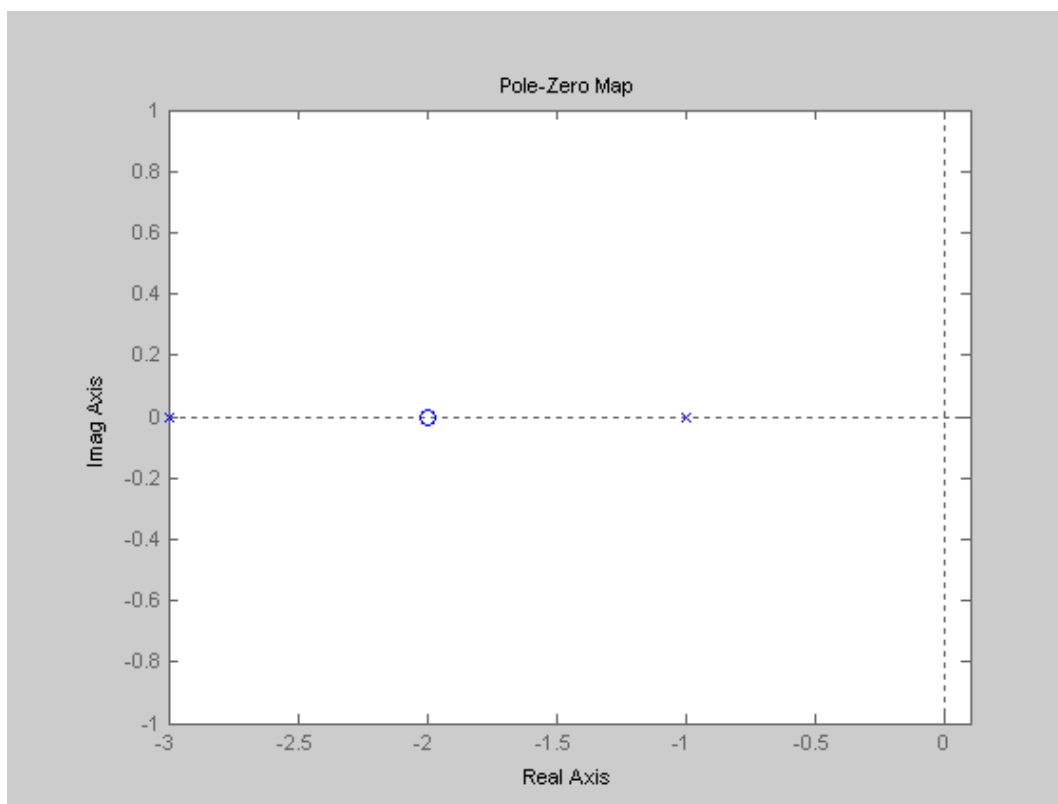
Zero 1 at -2 ($z_1 = 2$),

Pole 1 at -1 ($p_1 = 1$),

and

Pole 2 at -3 ($p_2 = 3$).

Conventionally, one uses a \times for a pole and a \circ for a zero. The pole-zero map for the given system can be located on the s-plane as shown below.



Example 6-3: Consider the characteristic equation of the closed-loop system listed below,

$$s(s+2)(s+3) + K(s+1) = 0$$

Locate the poles and zero of the open-loop transfer function.

Solution:

Similar to Example 6-1, one can divide both sides by $s(s+2)(s+3)$ and obtain the characteristic equation as follows:

$$1 + \frac{K(s+1)}{s(s+2)(s+3)} = 0$$

Therefore, the open-loop transfer function can be obtained,

$$G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)}$$

Now it is clear that there are one zero and three poles for the open-loop transfer function, namely,

Zero 1 at -1 ($z_1 = 1$),

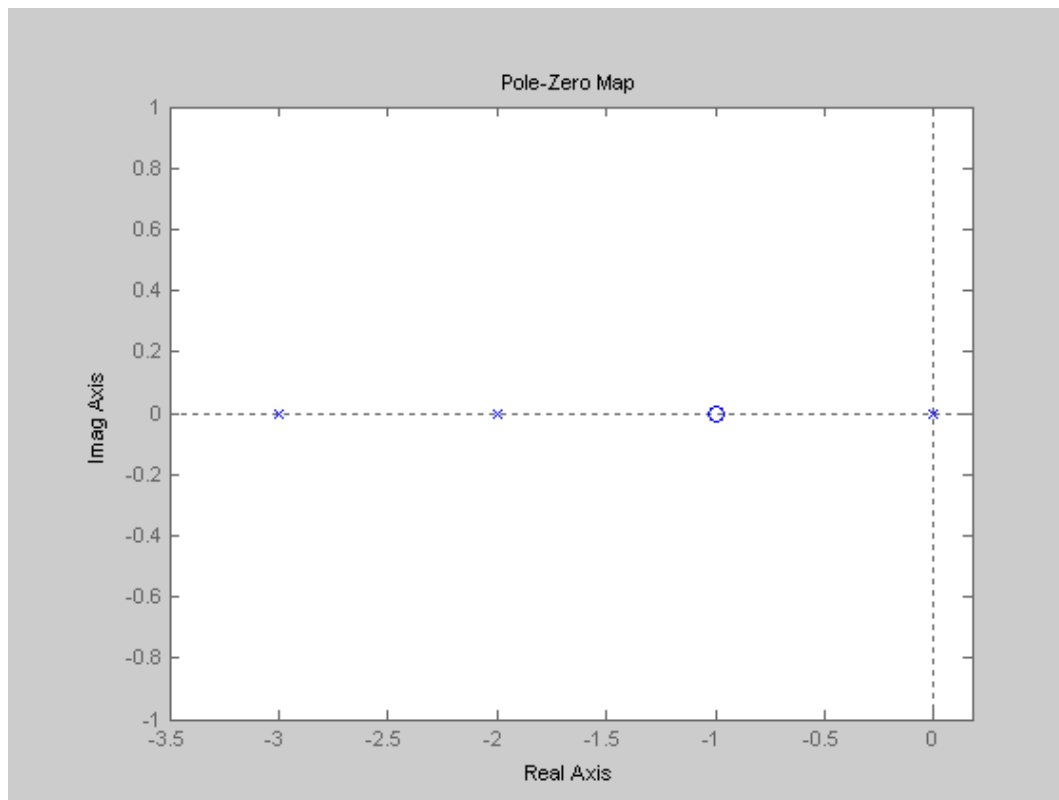
Pole 1 at 0 ($p_1 = 0$),

Pole 2 at -2 ($p_2 = 2$),

and

Pole 3 at -3 ($p_3 = 3$).

The pole-zero map for the given system can be located on the s-plane as shown below.



Number of Branches:

As one may expect, the number of branches of the Root Locus is the number of poles of the open-loop transfer function of the given system. Consequently, the number

of branches equals to the order of the polynomial $D(s)$, n . All the branches start at poles ($K = 0$) and approach zeros ($K = \infty$) of the open-loop transfer function. If there are more poles than zeros ($n - m > 0$), then, $n - m$ branches go to ∞ . As mentioned earlier, this course only discusses the Root Locus for $K > 0$.

Root Locus on the Real Axis:

The Root Locus on a given section of the Real Axis can be found in the section only if the total number of poles and zeros to the right of the section is an odd number. For the system given in Example 6-2, its Root Locus will be in the section between -1 and -2 since there is no pole or zero (total of 2, an even number) to the right of -1 , but there is one pole (at -1) to the right of -2 . Similarly, one can expect there is no RL in the section between -2 and -3 since there are one pole and one zero to the right of -3 . Finally, one can expect RL is in the section on the left of -3 since there are a total of three poles/zeros (two poles and one zero for this system) to the right of this section. To summarize the discussion, Figure 6-1 shows the sections where RL on the real axis. By analogy, Figure 6-2 shows the sections where RL on the real axis for the system given in Example 6-3.

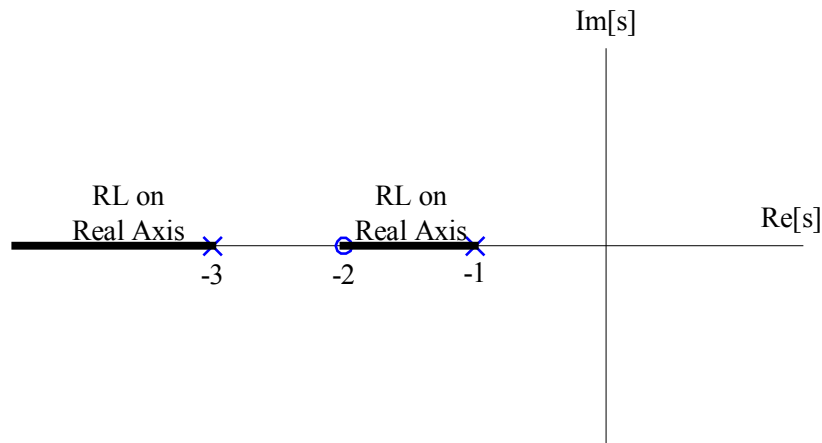


Fig. 6-1. Sections of real axis with Root Locus for the system given in Example 6-2.

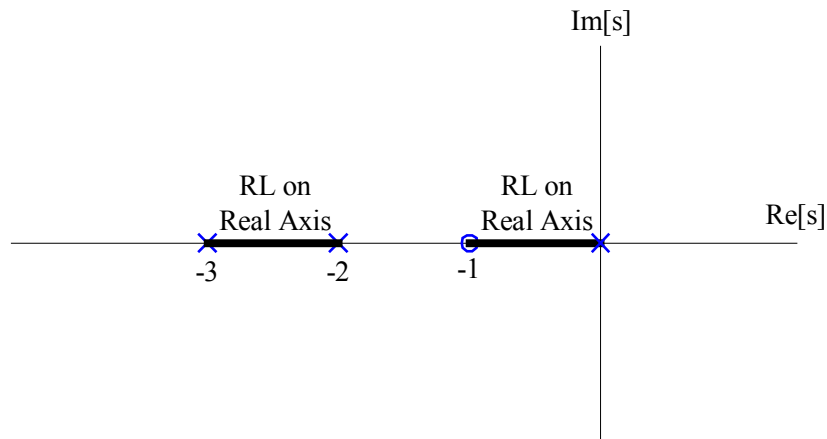


Fig. 6-2. Sections of real axis with Root Locus for the system given in Example 6-3.

Asymptote(s) of Root Locus:

The properties of the Root Locus when $K \rightarrow \infty$ in the s-plane are important to know. If $n - m > 0$, $n - m$ branches are approaching to infinity in the s-plane when $K \rightarrow \infty$. In the other words, for large distances from the origin in the s-plane, the branches of Root Locus approach a set of straight-line asymptotes. For instance, a system with $n = 3$ (the number of poles) and $m = 2$ (the number of zeros), then, as K increases, the Root Locus of the two poles would approach to the two zeros while the third pole would approach to the negative infinite of the real axis. In this case, there is only one ($n - m = 1$) asymptote. Another instance, a system with $n = 2$ and $m = 0$ (2 poles and no zero), as K increases, the root locus of the poles would approach the positive and the negative infinite of the imaginary axis. In this case, there are two ($n - m = 2$) asymptotes. These asymptotes from a point in the s-plane on the real axis called the center of asymptotes, or Centroid. The Centroid can be calculated as the sum of poles and zeros divided by $n - m$, namely,

$$\alpha = \frac{\sum \text{poles} - \sum \text{zeros}}{n - m} = - \frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n - m} \quad (6-10)$$

The angle between the asymptotes and the real axis can be calculated as follows:

$$\phi_A = \frac{\pi + 2k\pi}{n - m} \text{ rad, } k = 0, 1, 2, \dots, n - m \quad (K > 0) \quad (6-11)$$

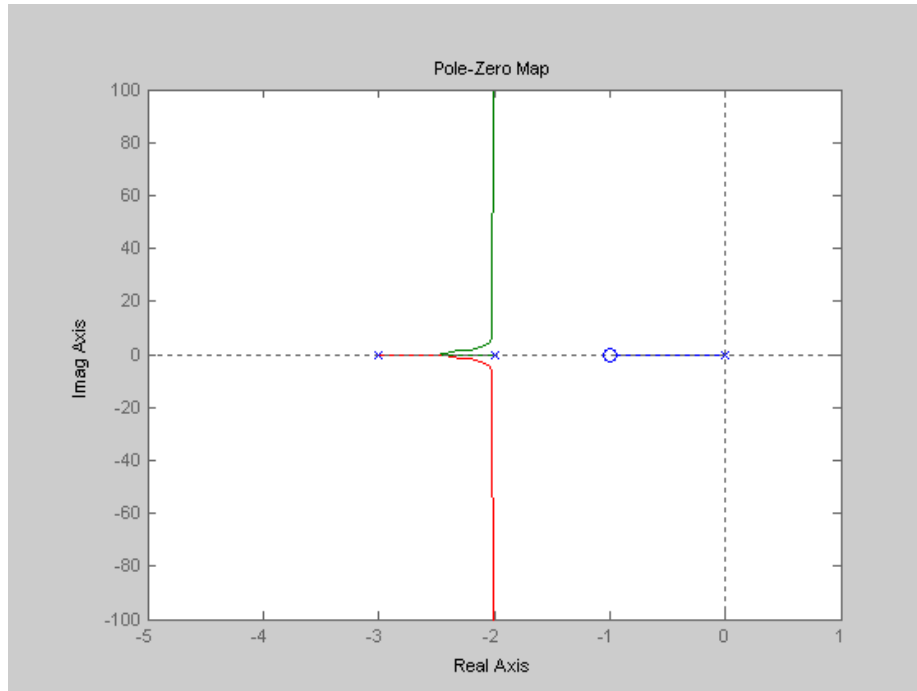
Example 6-4: What is the Centroid of the system given in Example 6-3?

Solution:

There are three poles ($p_1 = 0, p_2 = 2$ and $p_3 = 3$) and one zero ($z_1 = 1$) for the given system.

$$\alpha = - \frac{(0 + 2 + 3) - (1)}{3 - 1} = - \frac{4}{2} = -2$$

The Root Locus for the given system can be obtained with Matlab as shown below. As one can see clearly, when K increases the pole at 0 approaches to the zero at -1 while the other two poles approach to the two asymptotes with its Centroid at -2 . The angles of asymptotes are $\pi/2$ and $-\pi/2$ radians, or 90° and -90° , respectively.



As one may expect, the poles and zeros are not necessary located on the real axis only. For some poles and zeros, they may be complex conjugate pairs.

Angles of Departure (from Poles) and Angles of Arrival (at Zeros):

The angle of departure (arrival) of the RL at a pole (zero) of $G(s)H(s)$ denotes the behavior of the RL near that pole (zero). Equation (6-7) can be expanded as follows:

$$\angle G(s)H(s) = \sum_{i=1}^m \angle(s + z_i) - \sum_{j=1}^n \angle(s + p_j) = (2k + 1)\pi \tag{6-12}$$

where $k=0, \pm 1, \pm 2, \pm 3, \dots$. Since the direction of travel of the Root Locus departing from the pole(s) and arriving to the zero(s), the angle(s) of departure associated with the pole(s) and the angle(s) of arrival associated with the zero(s) are important. The angle of departure of the locus from a pole and the angle of arrival of the locus at a zero can be determined from the phase criterion. The angle of locus departure from a pole is the difference between the net angle due to all other poles and zeros and the criterion angle of $(2k + 1)\pi$, or simply π (or 180°), and similarly for the locus angle of arrival at zero. The angle of departure (or arrival) is particularly of interest for complex poles (and zeros) because the information is helpful in completing the Root Locus.

To find the angle of departure for a particular pole, for instance, p_1 , pick $s \rightarrow -p_1$, then, the angle of departure for p_1 , θ_{dp_1} , can be calculated as the angle of the open-loop transfer function, $\theta_{dp_1} = 180^\circ + \angle G(-p_1)H(-p_1)$, ignoring the contribution from the pole at $-p_1$. Namely,

$$\angle(-p_1 + z_1) + \angle(-p_1 + z_2) + \cdots + \angle(-p_1 + z_m) - (\theta_{dp_1} + \angle(-p_1 + p_2) + \cdots + \angle(-p_1 + p_n)) = 180^\circ$$

To find the angle of arrival for a particular zero, for instance, z_1 , pick $s \rightarrow -z_1$, then, the angle of arrival for z_1 , θ_{az_1} , can be calculated as the angle of the open-loop transfer function, $\theta_{az_1} = 180^\circ - \angle G(-z_1)H(-z_1)$, ignoring the contribution from $-z_1$. Namely,

$$\theta_{az_1} + \angle(-z_1 + z_2) + \cdots + \angle(-z_1 + z_m) - (\angle(-z_1 + p_1) + \angle(-z_1 + p_2) + \cdots + \angle(-z_1 + p_n)) = 180^\circ$$

Example 6-5: Consider a closed-loop system with an open-loop transfer function

$$G(s)H(s) = \frac{K(s+3-4j)(s+3+4j)}{(s+1)(s+3-2j)(s+3+2j)}$$

which has two zeros at $-3+4j$ (i.e., $z_1 = 3-4j$) and $-3-4j$ (i.e., $z_2 = 3+4j$), and three poles at -1 (i.e., $p_1 = 1$), $-3+2j$ (i.e., $p_2 = 3-2j$), and $-3-2j$ (i.e., $p_3 = 3+2j$). Find the angles of departure for p_1 , p_2 and p_3 , and the angles of arrival for z_1 and z_2 .

Solution:

The angle of departure from the pole at -1 (i.e., $p_1 = 1$):

Since there are three poles and two zeros, the Root Locus would have three branches with two branches starting from two of the complex poles and arriving at the two complex zeros while the pole on the real axis, p_1 , approaches to the infinity of the negative Real Axis. Therefore, the angle of departure for p_1 is 180° . Or, it could be calculated as follows:

$$\angle(-1+3-4j) + \angle(-1+3+4j) - (\theta_{dp_1} + \angle(-1+3-2j) + \angle(-1+3+2j)) = 180^\circ$$

$$\angle(2-4j) + \angle(2+4j) - (\theta_{dp_1} + \angle(2-2j) + \angle(2+2j)) = 180^\circ$$

$$-63.4^\circ + 63.4^\circ - (\theta_{dp_1} + (-45^\circ) + 45^\circ) = 180^\circ$$

$$\theta_{dp_1} = -180^\circ = 180^\circ$$

which gives the same answer.

The angle of departure from the pole at $-3+2j$ (i.e., $p_2 = 3-2j$):

$$\angle(-3+2j+3-4j) + \angle(-3+2j+3+4j) - (\angle(-3+2j+1) + \theta_{dp_2} + \angle(-3+2j+3+2j)) = 180^\circ$$

$$\angle(-2j) + \angle(6j) - (\angle(-2+2j) + \theta_{dp_2} + \angle(4j)) = 180^\circ$$

$$-90^\circ + 90^\circ - (135^\circ + \theta_{dp_2} + 90^\circ) = 180^\circ$$

$$\theta_{dp_2} = -135^\circ - 90^\circ - 180^\circ = -405^\circ = -45^\circ$$

The angle of departure from the pole at $-3-2j$ (i.e., $p_3 = 3+2j$):

$$\angle(-3-2j+3-4j) + \angle(-3-2j+3+4j) - (\angle(-3-2j+1) + \angle(-3-2j+3-2j) + \theta_{dp_3}) = 180^\circ$$

$$\angle(-6j) + \angle(2j) - (\angle(-2-2j) + \angle(-4j) + \theta_{dp_3}) = 180^\circ$$

$$-90^\circ + 90^\circ - (-135^\circ - 90^\circ + \theta_{dp_3}) = 180^\circ$$

$$\theta_{dp_3} = 135^\circ + 90^\circ - 180^\circ = 45^\circ$$

The angle of arrival to the zero at $-3 + 4j$ (i.e., $z_1 = 3 - 4j$):

$$\theta_{az_1} + \angle(-3 + 4j + 3 + 4j) - (\angle(-3 + 4j + 1) + \angle(-3 + 4j + 3 - 2j) + \angle(-3 + 4j + 3 + 2j)) = 180^\circ$$

$$\theta_{az_1} + \angle(8j) - (\angle(-2 + 4j) + \angle(2j) + \angle(6j)) = 180^\circ$$

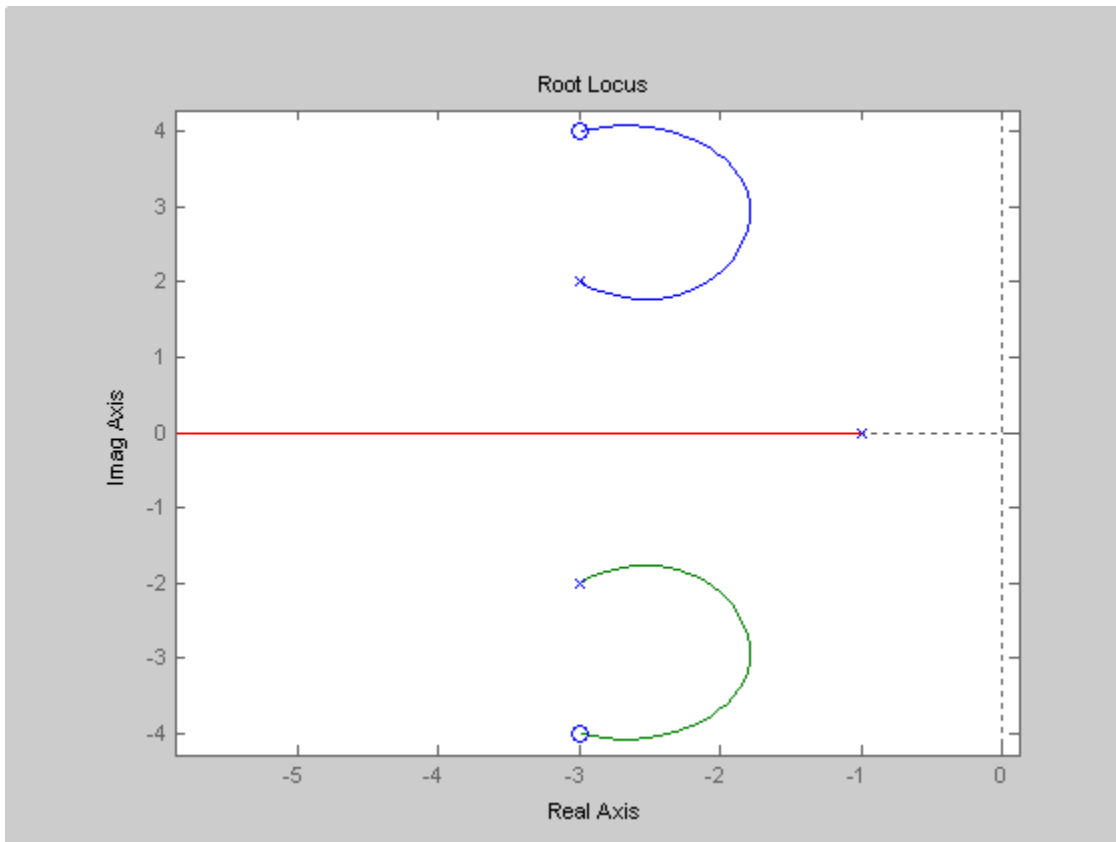
$$\theta_{az_1} = 180^\circ - 90^\circ + 116.6^\circ + 90^\circ + 90^\circ = 386.6^\circ = 26.6^\circ$$

The angle of arrival to the zero at $-3 - 4j$ (i.e., $z_2 = 3 + 4j$):

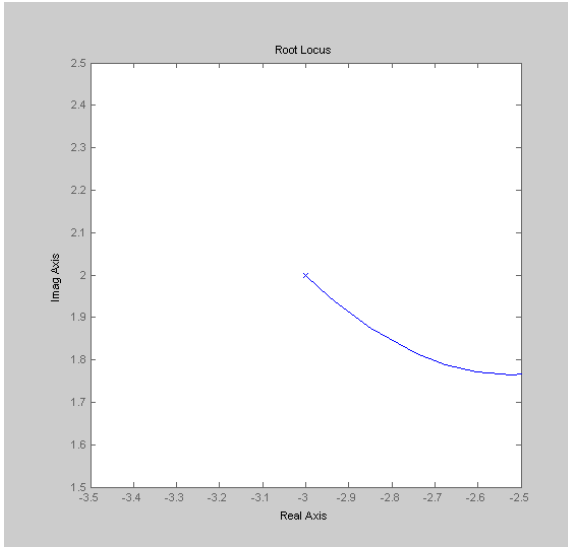
$$\angle(-3 - 4j + 3 - 4j) + \theta_{az_2} - (\angle(-3 - 4j + 1) + \angle(-3 - 4j + 3 - 2j) + \angle(-3 - 4j + 3 + 2j)) = 180^\circ$$

$$\angle(-8j) + \theta_{az_2} - (\angle(-2 - 4j) + \angle(-6j) + \angle(-2j)) = 180^\circ$$

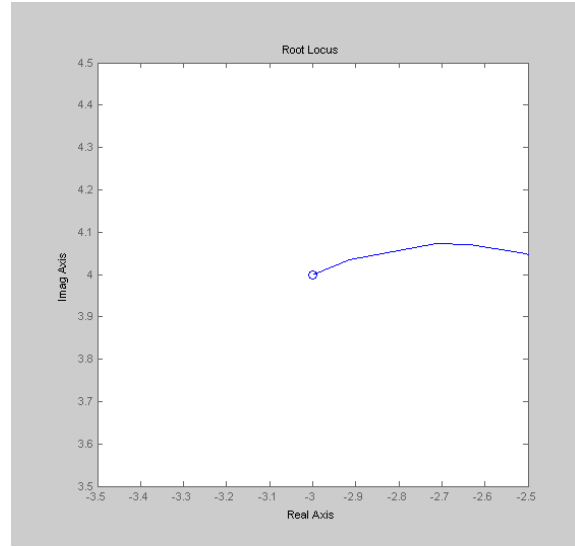
$$\theta_{az_2} = 180^\circ + 90^\circ + (-116.6^\circ) + (-90^\circ) + (-90^\circ) = -26.6^\circ$$



Root Locus for the given system showing the angles of departure and arrival



Angle of departure from $-3 + 2j$



Angle of arrival at $-3 + 4j$



Example 6-6: Consider a closed-loop system with an open-loop transfer function

$$G(s)H(s) = \frac{K(s+2)}{(s+1-j)(s+1+j)}$$

What are the angle of departure from the pole at $-1+j$ and the angle of arrival at the zero at -2 ?

Solution:

The angle of departure from the pole at $-1+j$ (i.e., $p_1 = 1-j$):

$$\angle(-1+j+2) - (\theta_{dp_1} + \angle(-1+j+1+j)) = 180^\circ$$

$$\angle(1+j) - (\theta_{dp_1} + \angle(2j)) = 180^\circ$$

$$\theta_{dp_1} = -180^\circ + 45^\circ - 90^\circ = -225^\circ = 135^\circ$$

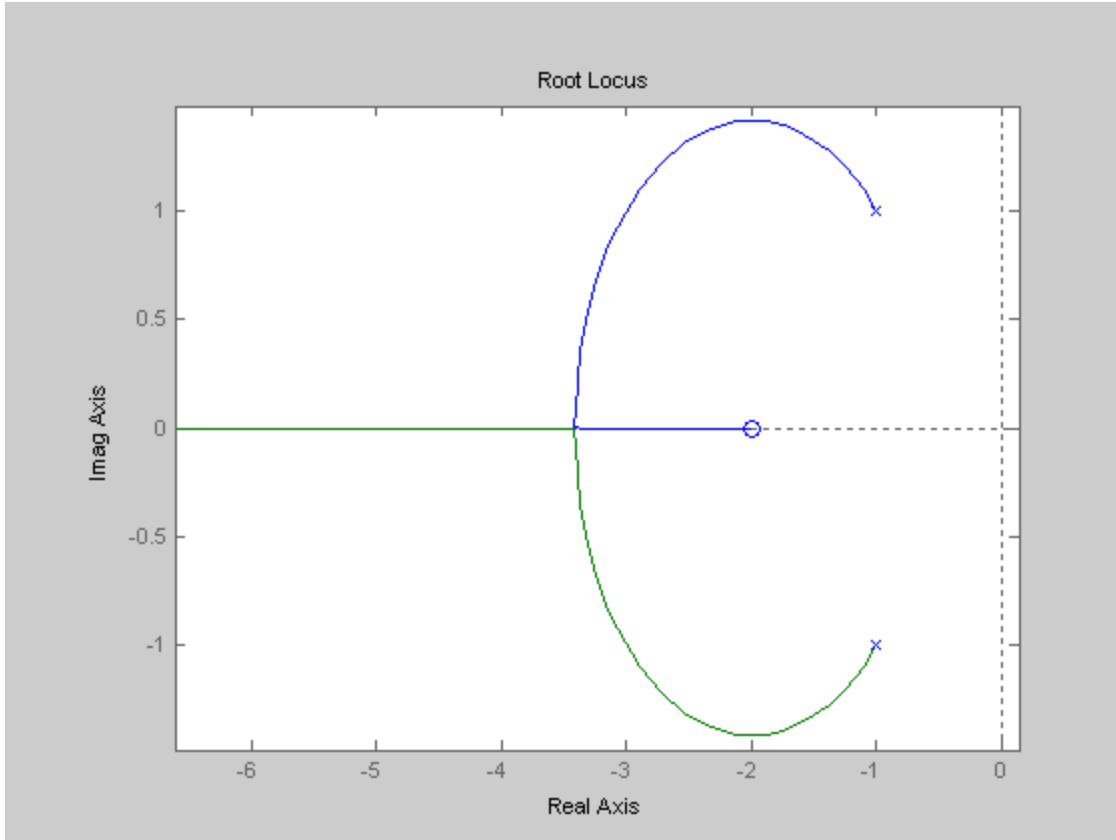
The angle of arrival at the zero -2 (i.e., $z_1 = 2$):

$$\theta_{az_1} - (\angle(-2+1-j) + \angle(-2+1+j)) = 180^\circ$$

$$\theta_{az_1} - (\angle(-1-j) + \angle(-1+j)) = 180^\circ$$

$$\theta_{az_1} = 180^\circ + (-135^\circ) + 135^\circ = 180^\circ$$

Those angles can be easily verified from the Root Locus shown below.



Crossing of the Imaginary Axis:

As discussed previously, some closed-loop system would be stable only with certain range of K, and that could be determined by Routh-Hurwitz Criterion. Also, when a system is going unstable from a stable region, its Root Locus would intersect with the Imaginary Axis (i.e., Root Locus traveling from the left-half-plane of s-plane to the right-half-plane). The point where the Root Locus intersects/crosses the Imaginary Axis of the s-plane, and the corresponding value of K, may be determined by the means of Routh-Hurwitz Criterion.

Example 6-7: Consider a closed-loop system with an open-loop transfer function

$$G(s)H(s) = \frac{K}{s(s+3)(s^2+2s+2)}$$

Where the Root Locus intersects the Imaginary Axis and at what K value?

Solution:

The characteristic equation: $s(s+3)(s^2+2s+2) + K = s^4 + 5s^3 + 8s^2 + 6s + K = 0$

Routh Table

$$\begin{array}{r} s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{array} \quad \begin{array}{r} 1 \quad 8 \quad K \\ 5 \quad 6 \\ b_1 \quad b_2 \\ c_1 \\ d_1 \end{array}$$

$$b_1 = \frac{5 \times 8 - 1 \times 6}{5} = \frac{34}{5} = 6.8$$

$$b_2 = \frac{5 \times K - 1 \times 0}{5} = K$$

$$c_1 = \frac{6.8 \times 6 - 5 \times K}{6.8} = \frac{40.8 - 5K}{6.8}$$

$$d_1 = \frac{c_1 \times b_2}{c_1} = b_2 = K$$

To have a stable system, $K > 0$ and $40.8 - 5K > 0$. Therefore, $0 < K < 8.16$.

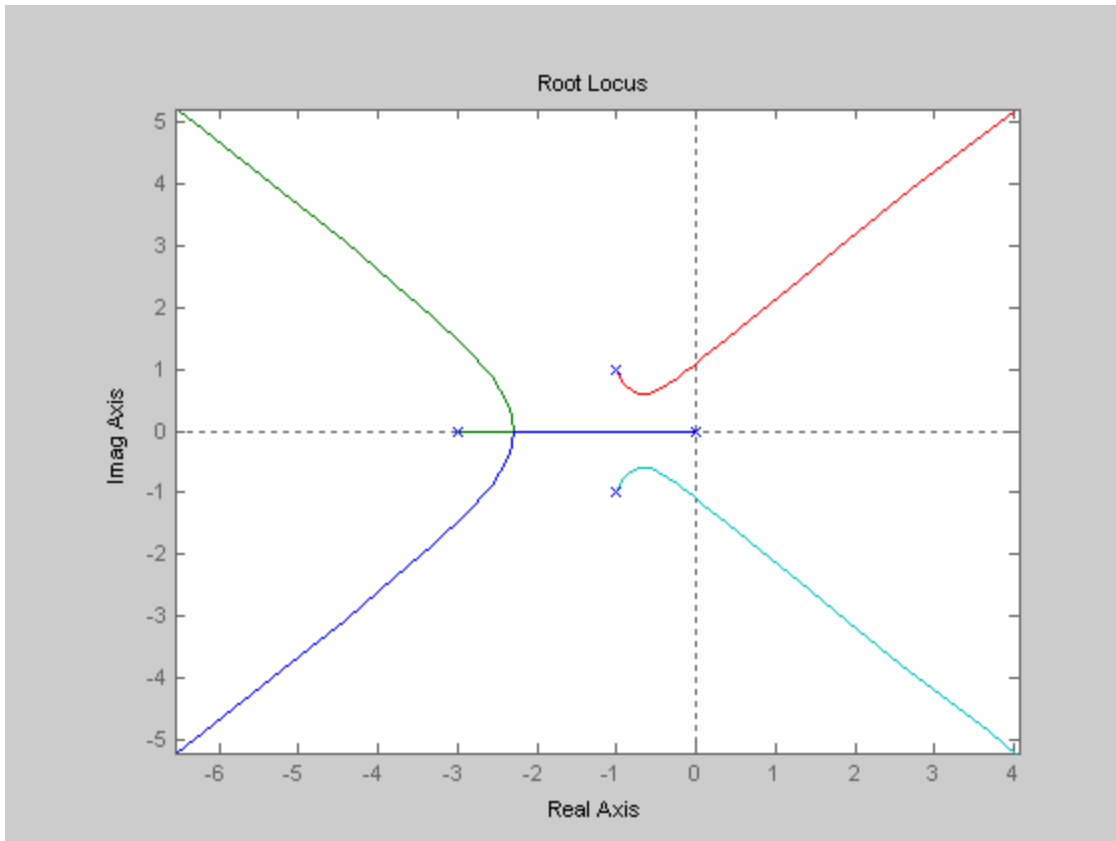
When $K=8.16$, the Root Locus intersects the Imaginary Axis which can be obtained from the equation starting with s^2 term in the Routh Table, namely, $b_1 s^2 + b_2 = 0$ with $K=8.16$.

$$6.8s^2 + 8.16 = 0$$

$$(s - j1.0954)(s + j1.0954) = 0$$

Therefore, the two branches of the Root Locus intersect the following two conjugate points at ± 1.0954 rad/sec.

The Root Locus for the given system is shown below.



Breakaway Points:

A Breakaway Point is a point on the Real Axis where two or more branches of the Root Locus depart from or arrive at the Real Axis. Figures 6-3 and 6-4 show two branches leaving and coming onto the Real Axis, respectively.

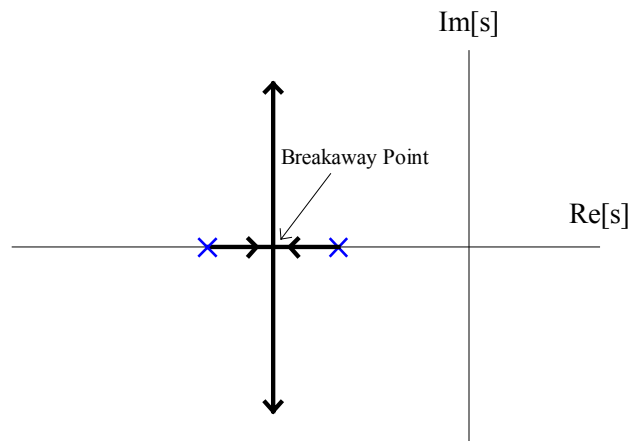


Fig. 6-3. Breakaway Point with two branches leaving the Real Axis.

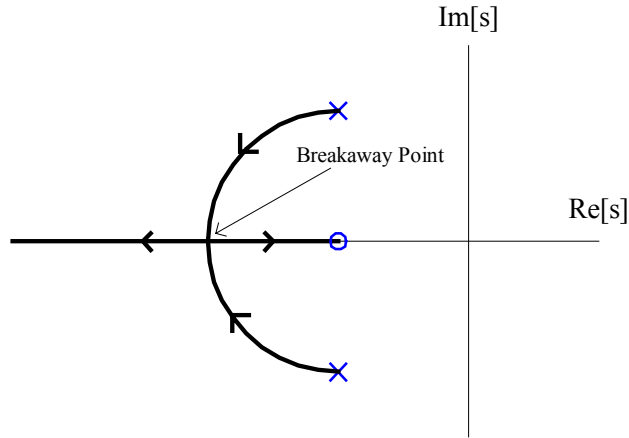


Fig. 6-4. Breakaway Point with two branches coming onto the Real Axis.

The Breakaway Point on the Real Axis can be evaluated graphically or analytically. Consider a unity feedback closed-loop system with an open-loop transfer function

$$G(s)H(s) = \frac{K}{(s+2)(s+4)}$$

The Root Locus is shown in Figure 6-5, and the Breakaway Point is at -3 .

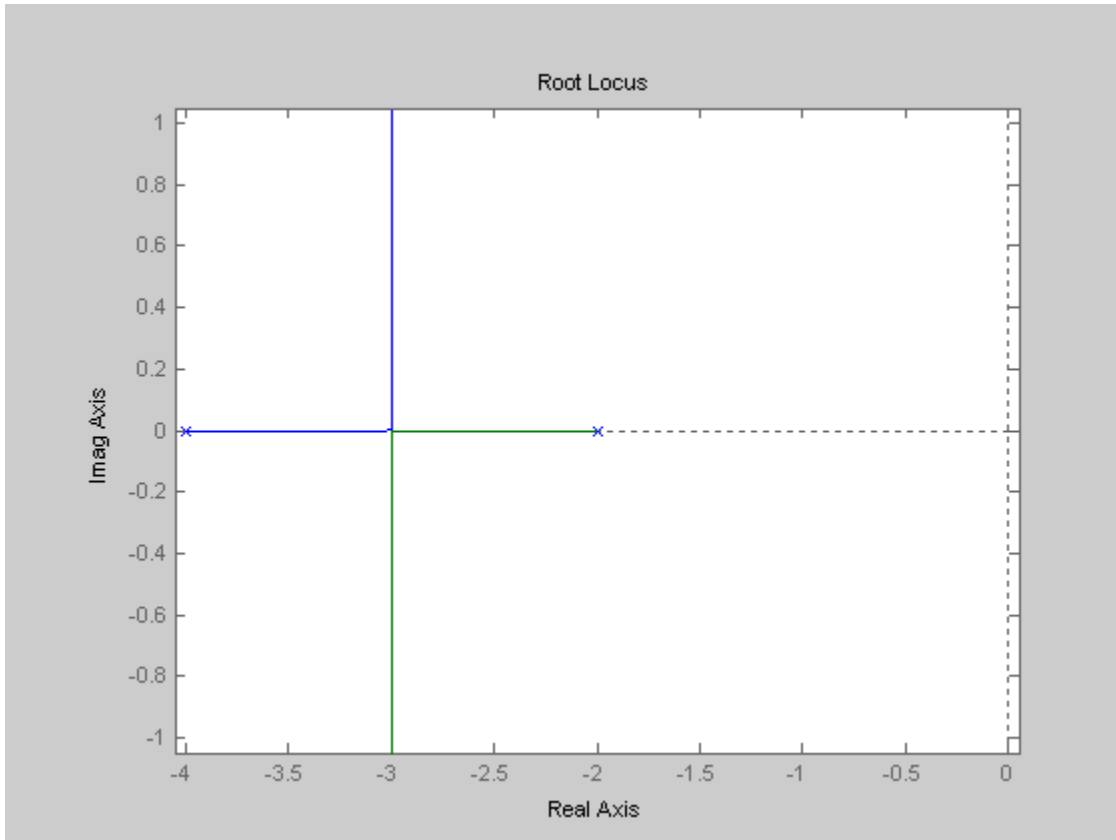


Fig. 6-5. Root Locus of system with open-loop transfer function $G(s)H(s) = \frac{K}{(s+2)(s+4)}$.

The Breakaway Point(s) of the Root Locus of $1 + G(s)H(s) = 1 + KG_1(s)H_1(s) = 0$, or

$$K = -\frac{1}{G_1(s)H_1(s)} \quad (6-13)$$

must satisfy

$$\frac{dK}{ds} = 0 \quad (6-14)$$

For the given system, $K = -\frac{1}{G_1(s)H_1(s)} = -(s+2)(s+4)$. Therefore,

$$\frac{dK}{ds} = -\frac{d(s^2 + 6s + 8)}{ds} = -(2s + 6) = 0$$

or

$s = -3$ is the Breakaway Point, same as obtained graphically from Figure 6-5.

It is important to point out that the condition for the Breakaway Point given by Equation (6-14) is necessary but not sufficient condition. In other words, all breakaway Point(s) must satisfy Equation (6-14), but not all solutions of Equation (6-14) are Breakaway Points.

Example 6-8: For the same system as given in Example 6-7, find the Breakaway Point.

Solution:

The open-loop transfer function is $G(s)H(s) = \frac{K}{s(s+3)(s^2+2s+2)}$.

Therefore, the K can be expressed as

$$K = -\frac{1}{G_1(s)H_1(s)} = -\frac{1}{\cancel{s(s+3)}(s^2+2s+2)} = -s(s+3)(s^2+2s+2) = -(s^4 + 5s^3 + 8s^2 + 6s)$$

$$\frac{dK}{ds} = \frac{d(s^4 + 5s^3 + 8s^2 + 6s)}{ds} = 4s^3 + 15s^2 + 16s + 6 = 0$$

Therefore, $s = -2.2886$, $s = -0.7307 + j0.3486$ and $s = -0.7307 - j0.3486$. However, the Breakaway Point is located only at -2.2886 where the Root Locus on that section of the Real Axis, as shown in the Root Locus for Example 6-7.

◆

Example 6-9: For the same system as given in Example 6-6, find the Breakaway Point.

Solution:

The open-loop transfer function is $G(s)H(s) = \frac{K(s+2)}{(s+1-j)(s+1+j)} = K \frac{(s+2)}{(s^2+2s+2)}$.

Therefore, the K can be expressed as

$$K = -\frac{1}{G_1(s)H_1(s)} = -\frac{1}{(s+2)/(s^2+2s+2)} = -\frac{(s^2+2s+2)}{(s+2)} = -(s^2+2s+2)(s+2)^{-1}$$

$$\frac{dK}{ds} = \frac{d((s^2+2s+2)(s+2)^{-1})}{ds} = (2s+2)(s+2)^{-1} + (-1)(s+2)^{-2}(s^2+2s+2) = 0$$

$$\frac{(2s+2)}{(s+2)} - \frac{s^2+2s+2}{(s+2)^2} = \frac{(2s+2)(s+2) - s^2 - 2s - 2}{(s+2)^2} = \frac{s^2+4s+2}{(s+2)^2} = 0$$

$$s^2+4s+2=0$$

Therefore, $s = -3.4142$ and $s = -0.5858$. However, the Breakaway Point is located only at -3.4142 where the Root Locus on that section of the Real Axis, as shown in the Root Locus for Example 6-6. ♦

Symmetry of the Root Locus:

Another important property of the Root Locus is that a Root Locus is symmetrical with respect to the Real Axis of the s-plane. In general, the Root Locus is symmetrical with respect to the axes of symmetry of the poles and zeros of $G(s)H(s)$. One should observe such a property from the plots given throughout in this course.

VII. Guidelines for Sketching a Root Locus

The details of the basic properties of the Root Locus of the closed-loop systems have been discussed in the previous section. In this section, a set of step-by-step guidelines for sketching a Root Locus is summarized, and a detailed step-by-step example is presented to illustrate how to use those guidelines to sketch a Root Locus.

Rules for the Construction of the Root Locus:

- 1) Number of Branches: n , where n is the number of poles.
- 2) Direction of Travel: As $K \rightarrow \infty$, the Root Locus approaches zeros of open-loop transfer function. If $n - m > 0$ (m is the number of zeros), then, $n - m$ branches go to ∞ .
- 3) Root Locus on the Real Axis: If the numbers of poles plus zeros on the right hand side is an odd number, Root Locus is on the real axis. On the other word, draw Root Locus on the Real Axis to the left of an odd number of poles and zeros.

- 4) Number of Asymptotes, Angle of Asymptotes and Centroid:

Number of asymptotes: $n - m$

Angle of asymptote: The angle(s) between the real axis and the asymptote(s) can be calculated as

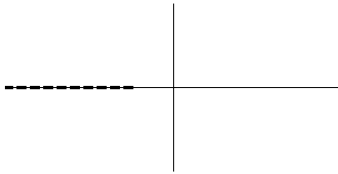

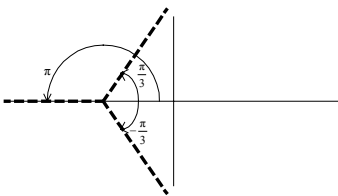
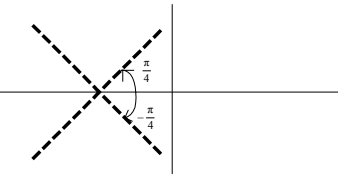
$$\phi_A = \frac{\pi + 2k\pi}{n - m}, k = 0, 1, 2, \dots, n - m \quad (K > 0)$$

Centroid:

$$\alpha = \frac{\sum \text{poles} - \sum \text{zeros}}{n - m} = - \frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n - m}$$

Table 6-1 shows the angle(s) of asymptote(s) for $n - m = 1, 2, 3$ and 4 , numerically (in radians) and graphically.

Table 6-1. Graphical representation of Asymptotes (for n-m =1, 2, 3 & 4).

n - m	ϕ_A	Graphical Representation of Asymptotes
1	π	
2	$\frac{\pi}{2}, -\frac{\pi}{2}$	
3	$\frac{\pi}{3}, \pi, -\frac{\pi}{3}$	
4	$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$	

6) Angles of Departure and Angles of Arrival:

Angle of departure, θ_{dp_1} :

$$\angle(-p_1 + z_1) + \angle(-p_1 + z_2) + \dots + \angle(-p_1 + z_m) - (\theta_{dp_1} + \angle(-p_1 + p_2) + \dots + \angle(-p_1 + p_n)) = 180^\circ$$

Angle of Arrival, θ_{az_1} :

$$\theta_{az_1} + \angle(-z_1 + z_2) + \dots + \angle(-z_1 + z_m) - (\angle(-z_1 + p_1) + \angle(-z_1 + p_2) + \dots + \angle(-z_1 + p_n)) = 180^\circ$$

7) Crossing of the Imaginary Axis: The locations of the Root Locus intersects the Imaginary Axis can be obtained from the equation starting with s^2 term in the Routh Table, namely, $b_1s^2 + b_2 = 0$ with K equals the maximum value of the stable range obtained by Routh-Hurwitz Criterion.

8) Breakaway Point(s): The breakaway point(s) of the Root Locus of $1 + G(s)H(s) = 1 + KG_1(s)H_1(s) = 0$, or $K = -\frac{1}{G_1(s)H_1(s)}$ must satisfy $\frac{dK}{ds} = 0$.

9) Root Locus is symmetry with respect to the Real Axis.

Example 6-10: A closed-loop negative feedback system has an open-loop transfer function of

$$G(s)H(s) = \frac{K(s+10)}{s(s^2 + 6s + 13)}$$

Sketch the Root Locus for $K > 0$.

Solution:

There are three poles and one zero, i.e., $n=3$ and $m=1$. Therefore, the Root Locus has three branches ($n=3$), one branch starting from one of the poles and approaching to the zero while the other two branches starting from the two ($n-m=3-1=2$) remaining poles and approaching to their Asymptotes ($n-m=2$) going to infinite when K increases.

The poles are located at 0 , $-3 + j2$ and $-3 - j2$ while the zero is located at -10 . The Root Locus is on the Real Axis between 0 and -10 since the number of pole(s) is an odd number on the right hand side of the zero at -10 .

The angle of Asymptotes can be calculated as follows

$$\phi_A = \frac{\pi + 2k\pi}{n - m} = \begin{cases} \frac{\pi}{3-1} = \frac{\pi}{2}, & \text{for } k = 0 \\ \frac{\pi + 2\pi}{3-1} = \frac{3\pi}{2} = -\frac{\pi}{2}, & \text{for } k = 1 \end{cases}$$

The Centroid can be obtained by

$$\alpha = \frac{\sum \text{poles} - \sum \text{zeros}}{n - m} = \frac{-\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n - m} = \frac{-(0 + (3 - 2j) + (3 + 2j)) - (-10)}{3 - 1} = 2$$

A Centroid with a positive value implies two branches of the Root Locus would travel from the left-half-plane to the right-half-plane of the s -plane. Therefore, those branches would cross the Imaginary Axis and the locations can be obtained as follows:

The characteristic equation: $s(s^2 + 6s + 13) + K(s + 10) = s^3 + 6s^2 + (13 + K)s + 10K = 0$

Routh Table

$$s^3 \quad 1 \quad 13 + K$$

$$s^2 \quad 6 \quad 10K$$

$$s^1 \quad b_1$$

$$s^0 \quad c_1$$

$$b_1 = \frac{6 \times (13 + K) - 1 \times 10K}{6} = \frac{78 - 4K}{6}$$

$$c_1 = 10K$$

To have a stable system, $K > 0$ and $78 - 4K > 0$. Therefore, $0 < K < 19.5$.

When $K=19.5$, the Root Locus intersects the Imaginary Axis which can be obtained from the equation starting with s^2 term in the Routh Table, namely, $6s^2 + 10K = 0$ with $K=19.5$.

$$6s^2 + 195 = 0$$

$$(s - j5.7009)(s + j5.7009) = 0$$

Therefore, the two branches of the Root Locus intersect the following two conjugate points at ± 5.7009 rad/sec with $K=19.5$.

The angle of departure from the pole at 0 (i.e., $p_1 = 0$):

$$\angle(0+10) - (\theta_{dp_1} + \angle(0+3-2j) + \angle(0+3+2j)) = 180^\circ$$

$$\angle(10) - (\theta_{dp_1} + \angle(3-2j) + \angle(3+2j)) = 180^\circ$$

$$0^\circ - (\theta_{dp_1} + 146.3^\circ + (-146.3^\circ)) = 180^\circ$$

$$\theta_{dp_1} = 0^\circ - 146.3^\circ - (-146.3^\circ) - 180^\circ = -180^\circ = 180^\circ$$

The angle of departure from the pole at $-3 + 2j$ (i.e., $p_2 = 3 - 2j$):

$$\angle(-3+2j+10) - (\angle(-3+2j+0) + \theta_{dp_2} + \angle(-3+2j+3+2j)) = 180^\circ$$

$$\angle(7+2j) - (\angle(-3+2j) + \theta_{dp_2} + \angle(4j)) = 180^\circ$$

$$\theta_{dp_2} = -180^\circ + 15.9^\circ - 146.3^\circ - 90^\circ = -400.4^\circ = -40.4^\circ$$

The angle of departure from the pole at $-3 - 2j$ (i.e., $p_3 = 3 + 2j$):

$$\angle(-3-2j+10) - (\angle(-3-2j+0) + \angle(-3-2j+3-2j) + \theta_{dp_3}) = 180^\circ$$

$$\angle(7-2j) - (\angle(-3-2j) + \angle(-4j) + \theta_{dp_3}) = 180^\circ$$

$$\theta_{dp_3} = -180^\circ - 15.9^\circ - (-146.3^\circ) - (-90^\circ) = 400.4^\circ = 40.4^\circ$$

The angle of arrival at the zero -10 (i.e., $z_1 = 10$)

$$\theta_{az_1} - (\angle(-10+0) + \angle(-10+3-2j) + \angle(-10+3+2j)) = 180^\circ$$

$$\theta_{az_1} - (\angle(-10) + \angle(-7-2j) + \angle(-7+2j)) = 180^\circ$$

$$\theta_{az_1} = 180^\circ + (180^\circ) + (-164.1^\circ) + 164.1^\circ = 360^\circ = 0^\circ$$

With all the calculations, one might be able to conclude that there is no Breakaway Point for the given system. One might pursue evaluating the Breakaway Point anyway and should conclude the same thing.

$$K = -\frac{1}{\frac{(s+10)}{s(s^2 + 6s + 13)}} = -\frac{s(s^2 + 6s + 13)}{(s+10)} = -(s^3 + 6s^2 + 13s)(s+10)^{-1}$$

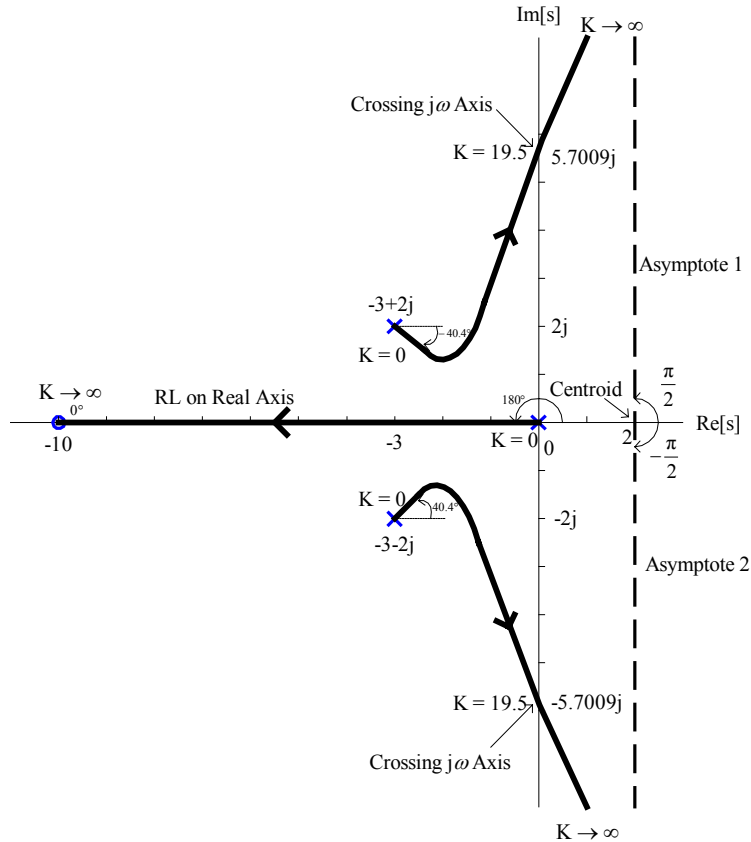
$$\frac{dK}{ds} = \frac{d((s^3 + 6s^2 + 13s)(s+10)^{-1})}{ds} = (3s^2 + 12s + 13)(s+10)^{-1} + (-1)(s+10)^{-2}(s^3 + 6s^2 + 13s) = 0$$

$$\frac{(3s^2 + 12s + 13)}{(s+10)} - \frac{s^3 + 6s^2 + 13s}{(s+10)^2} = \frac{(3s^2 + 12s + 13)(s+10) - s^3 - 6s^2 - 13s}{(s+10)^2} = \frac{2s^3 + 36s^2 + 120s + 130}{(s+10)^2} = 0$$

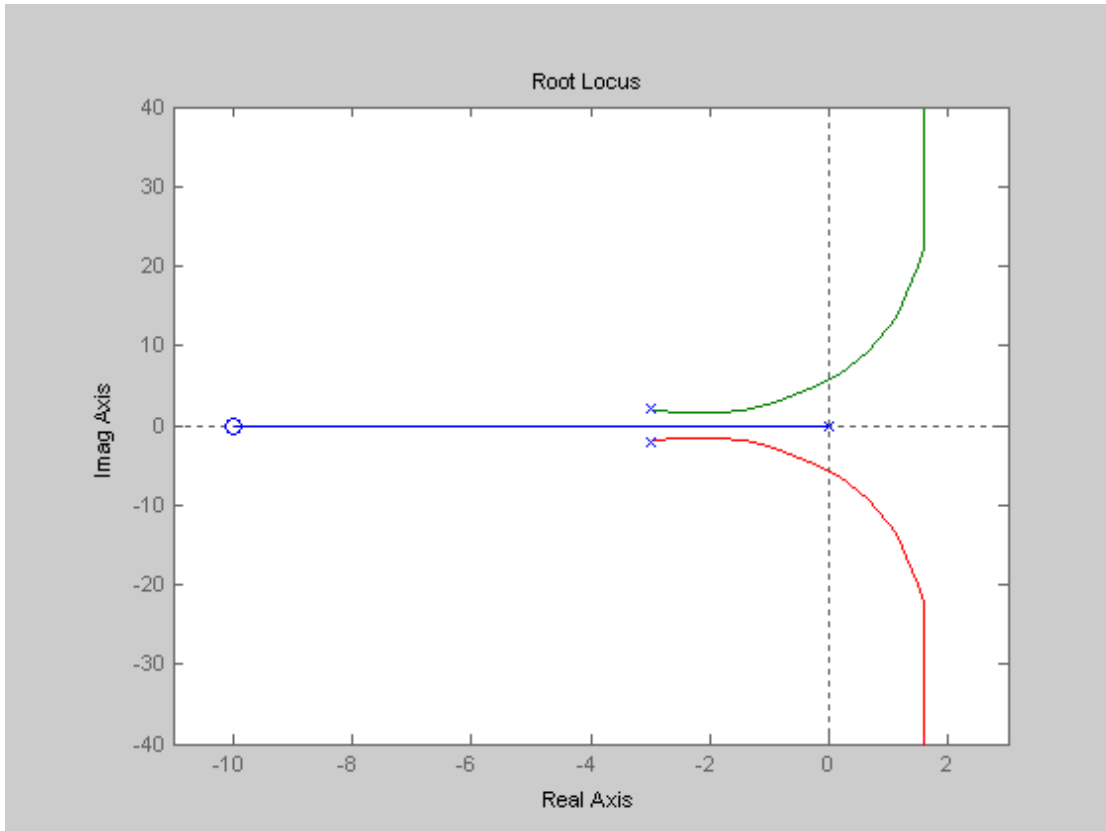
$$2s^3 + 36s^2 + 120s + 130 = s^3 + 18s^2 + 60s + 65 = 0$$

Therefore, $s = -1.9691 \pm 0.8632j$ and $s = -14.0619$. However, there is no Root Locus on the Real Axis at -14.0619 . So, no Breakaway Point exists as expected.

The Root Locus for the given system is sketched below using the information obtained from following the calculations provided in the Guidelines.

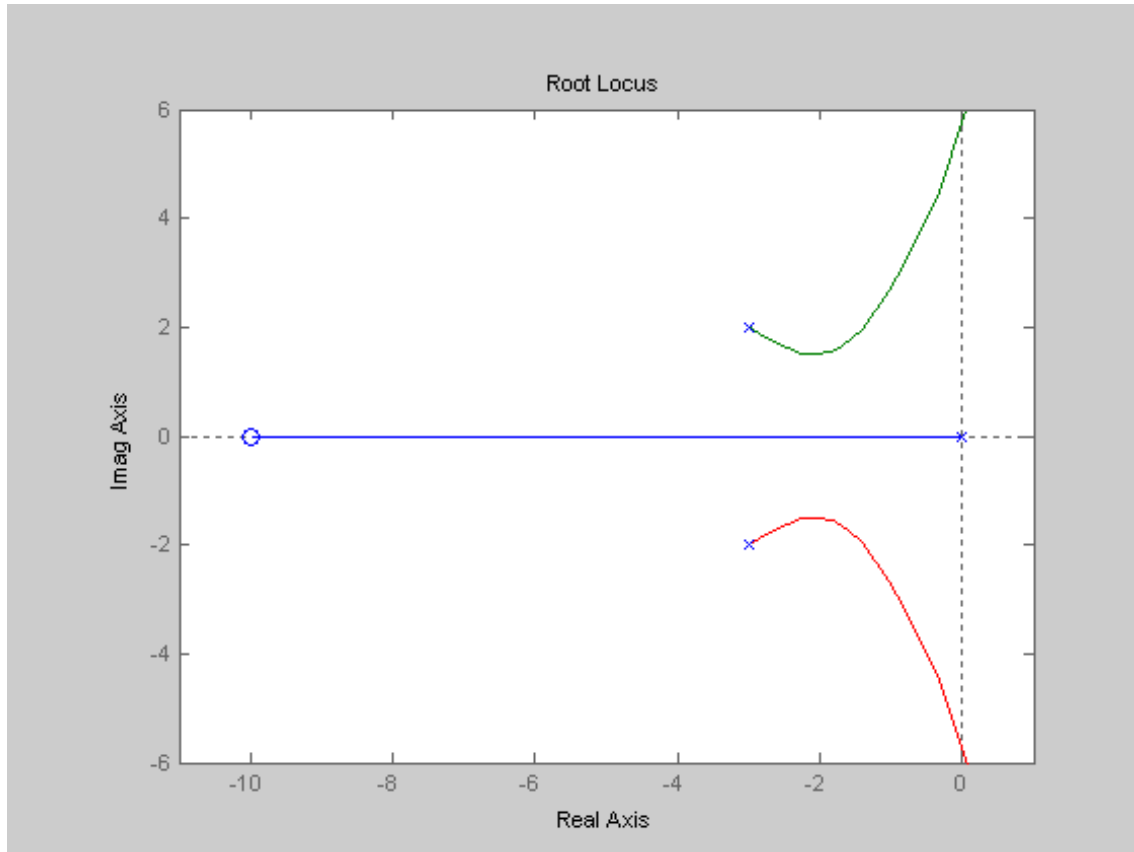


The Root Locus obtained by using MATLAB is shown below to verify the sketched Root Locus.



Root Locus of system with open-loop transfer function $G(s)H(s) = \frac{K(s+10)}{s(s^2+6s+13)}$.

To show the angles of departure and arrival and the locations where the two branches crossing the Imaginary Axis, the Root Locus of the same system with a modified scale is shown below.



Root Locus with a better scale to show the angles of departure and arrival and the locations where two branches crossing the Imaginary Axis.

Also, as one can easily observed that the Root Locus is symmetrical to the Real Axis.



References:

- [1] Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, *Feedback Control of Dynamic Systems – 2nd Edition*, Addison Wesley, 1991
- [2] Benjamin C. Kuo, *Automatic Control Systems – 5th Edition*, Prentice-Hall, 1987
- [3] Richard C. Dorf, *Modern Control Systems – 6th Edition*, Addison Wesley, 1992
- [4] John A. Camara, *Practice Problems for the Electrical and Computer Engineering PE Exam – 6th Edition*, Professional Publications, 2002
- [5] NCEES, *Fundamentals of Engineering Supplied-Reference Handbook – 6th Edition*, 2003
- [6] Merle C. Potter, *FE/EIT Electrical Discipline-Specific Review for the FE/EIT Exam – 5th Edition*, Great Lakes Press, 2001
- [7] Merle C. Potter, *Principles & Practice of Electrical Engineering – 1st Edition*, Great Lakes Press, 1998
- [8] Joseph J. DiStefano, III, Allen R. Stubberud, and Ivan J. Williams, *Feedback and Control Systems – 2nd Edition*, McGraw-Hill, 1990

Appendix A

How to use MATLAB[®] to obtain the Root Locus

Example 6-5:

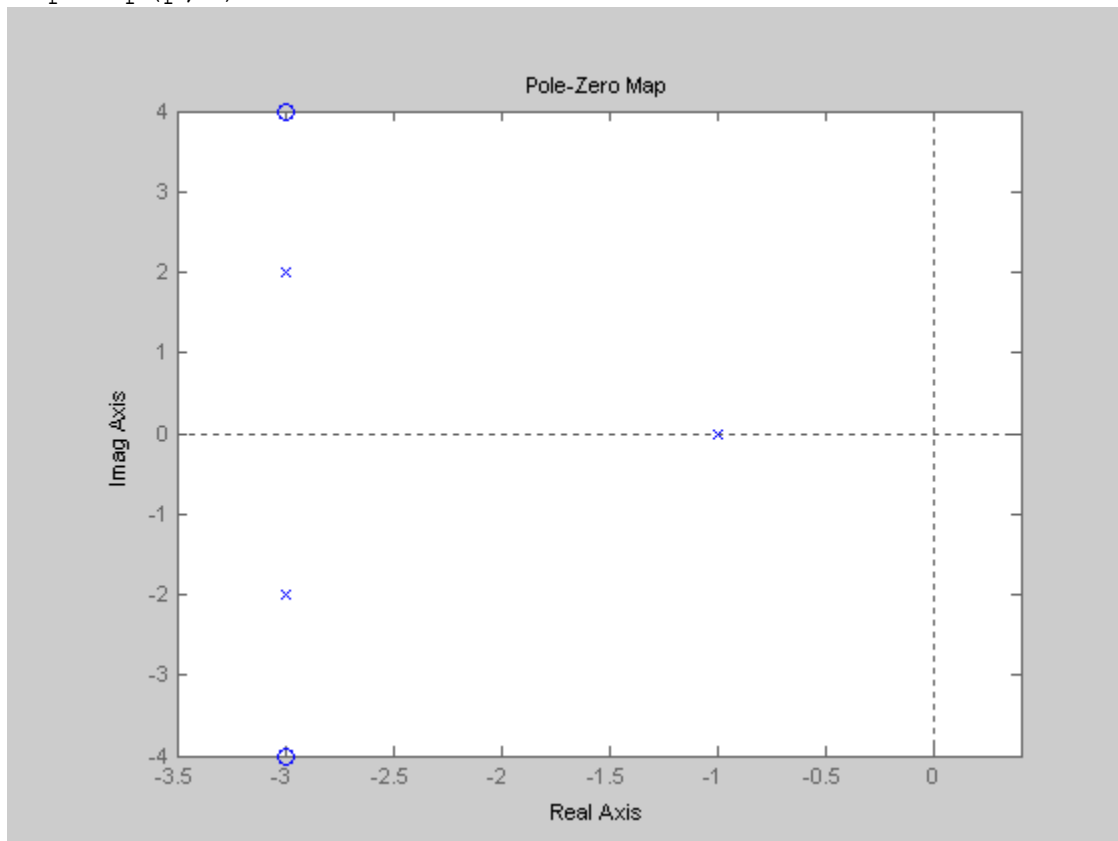
```
>> num=[1 6 25];  
>> den=[1 7 19 13];  
>> [z,p,k]=tf2zp(num,den);  
>> z
```

```
z =  
  
-3.0000 + 4.0000i  
-3.0000 - 4.0000i
```

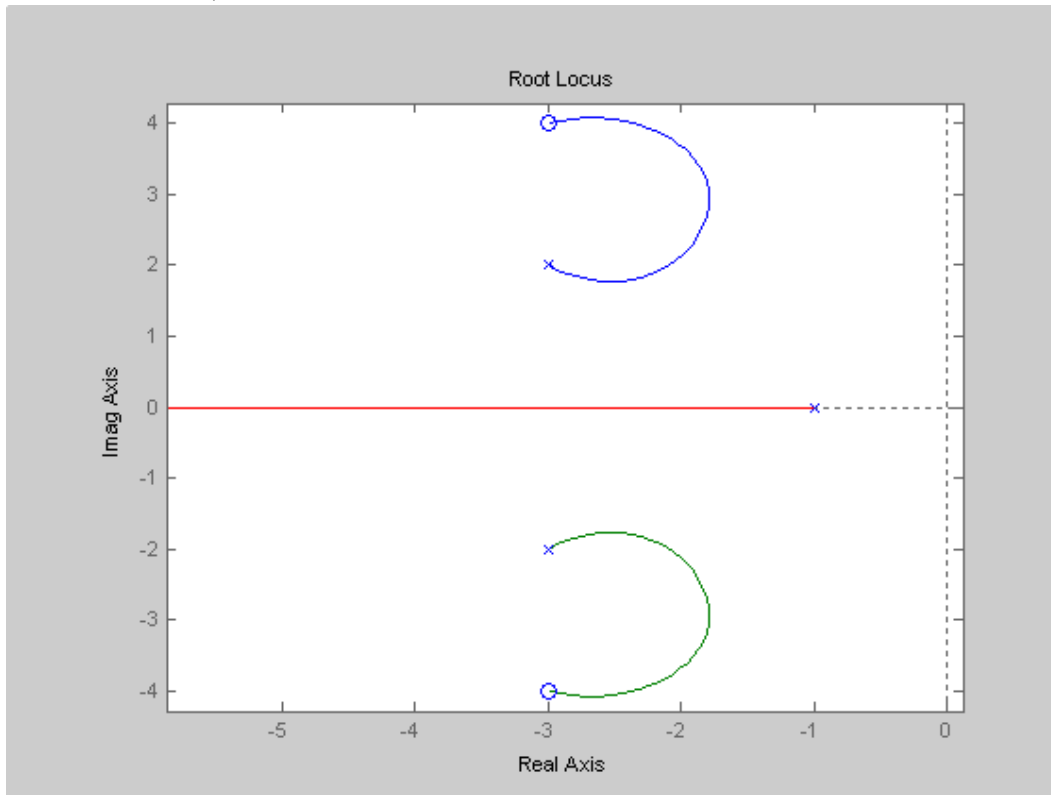
```
>> p
```

```
p =  
  
-3.0000 + 2.0000i  
-3.0000 - 2.0000i  
-1.0000
```

```
>> pzmap(p,z)
```

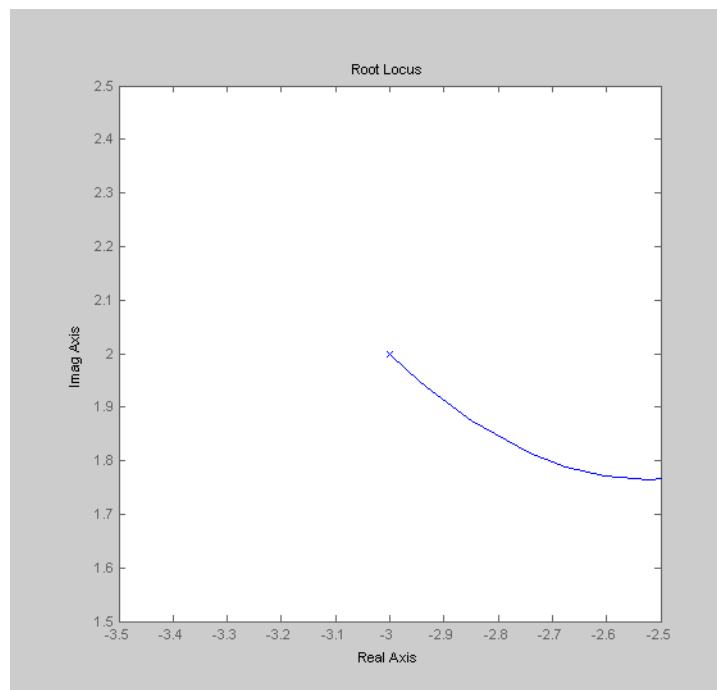


```
>> rlocus(num,den)
```

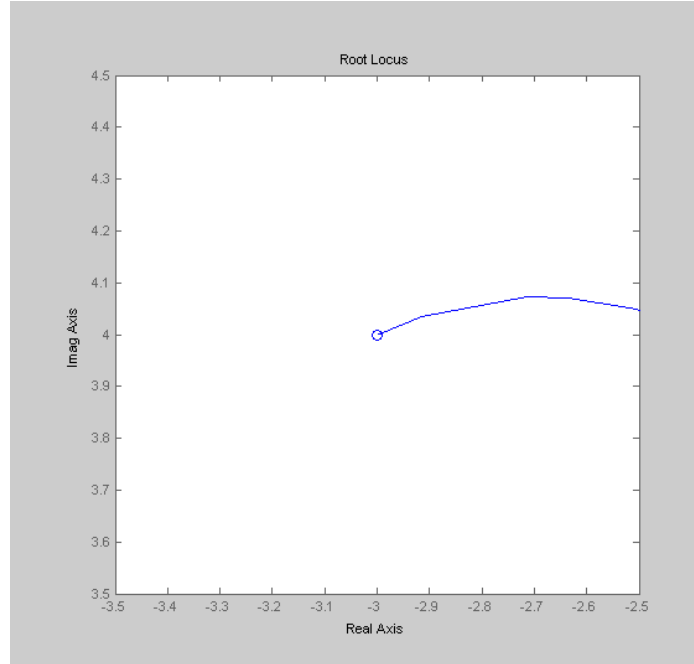


To change scale of the Real Axis and Imaginary Axis to show the angles of departure and arrival

```
>> v=[-3.5 -2.5 1.5 2.5];  
>> axis(v)
```

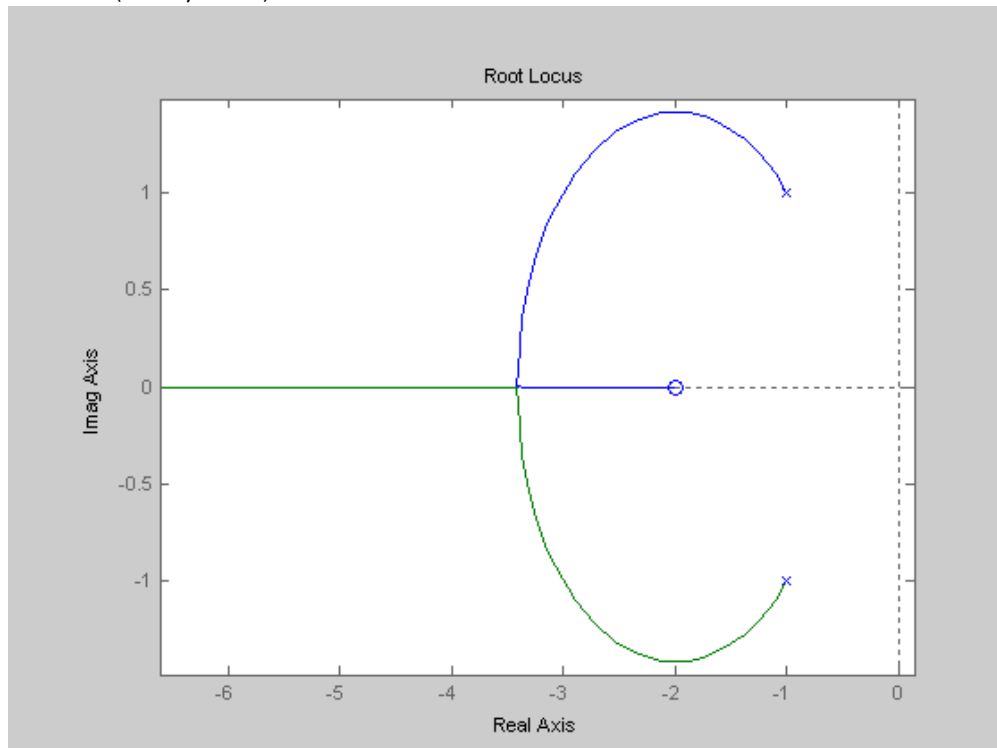


```
>> v=[-3.5 -2.5 3.5 4.5];  
>> axis(v)
```



Example 6-6:

```
>> num=[1 2];  
>> den=[1 2 2];  
>> rlocus(num,den)
```



Example 6-10:

```
>> num=[1 10];  
>> den=[1 6 13 0];  
>> [p,z]=pzmap(num,den)
```

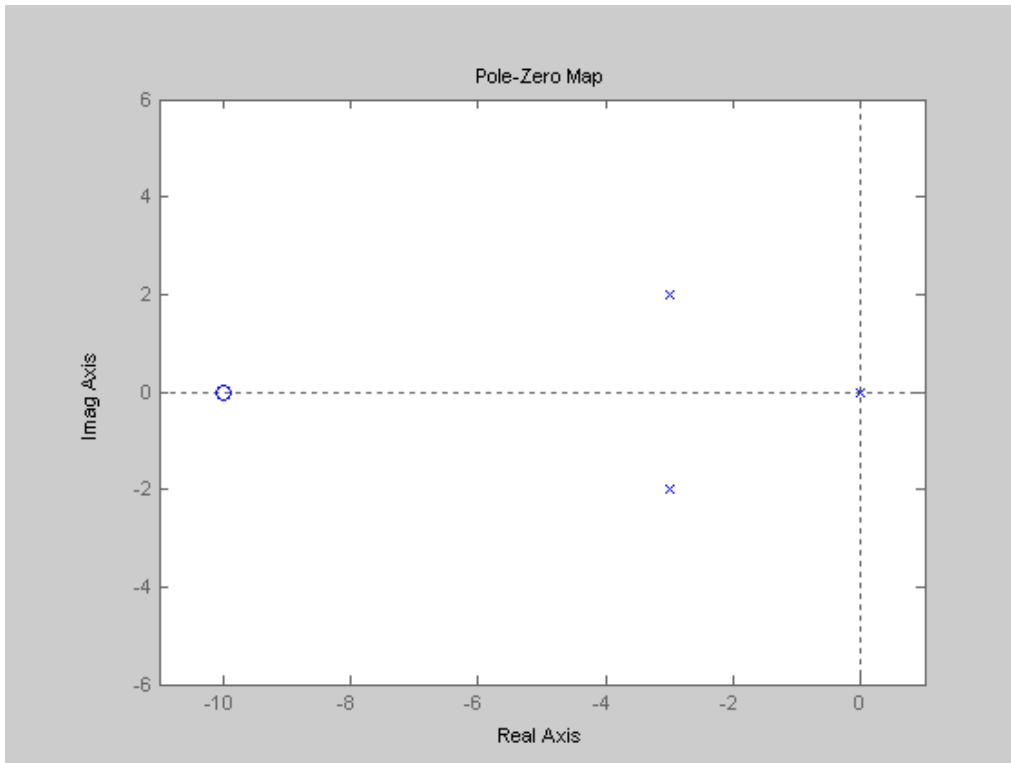
p =

```
0  
-3.0000 + 2.0000i  
-3.0000 - 2.0000i
```

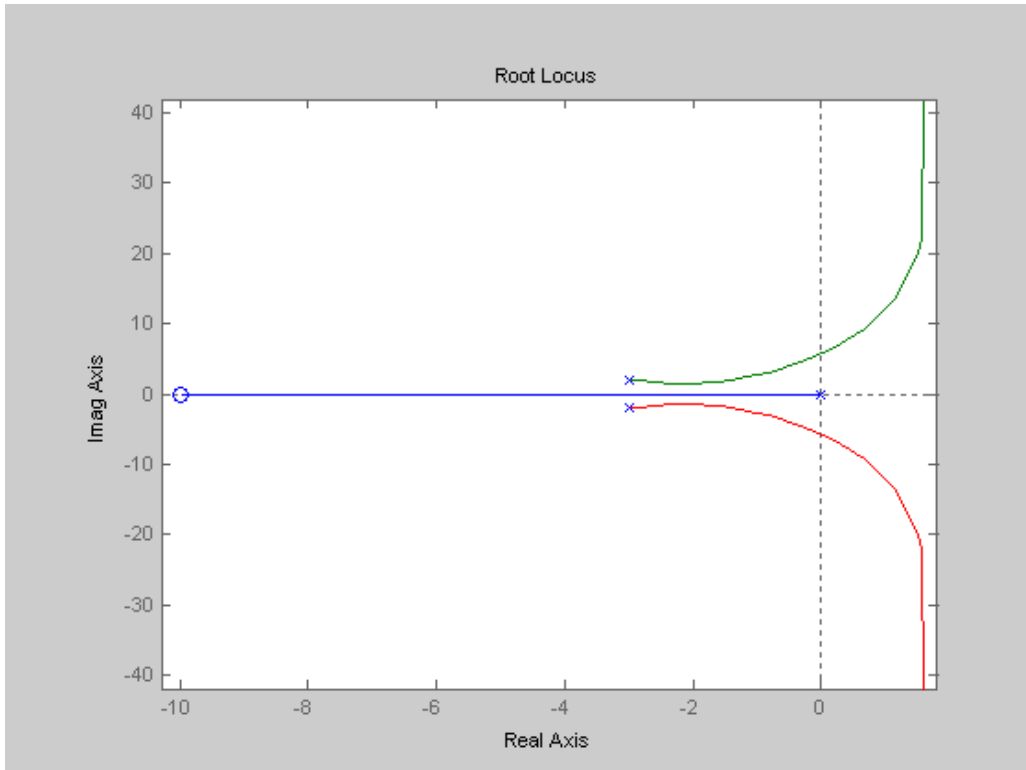
z =

```
-10
```

```
>> pzmap(p,z)  
>> v=[-11 1 -6 6];  
>> axis(v)
```



```
>> rlocus(num,den)
```



```
>> v=[-11 1 -6 6];  
>> axis(v)
```

