PDHonline Course G204 (6 PDH)

Design for Static Strength

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Design for Static Strength

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1.0 Prerequisites

It is assumed the student has a working knowledge of stress analysis and is able to determine the state of stress at the critical location of the part. It is also assumed that the materials being evaluated are linear elastic materials, and that the loading is static.

2.0 Material Properties and Testing

When a body is subjected to a purely axial load, it will elongate. This elongation is called *normal strain*, ε, and it is defined as the change in elongation, ΔL, divided by the original length, L. Thus normal engineering strain is defined as:

$$\varepsilon = \Delta L/L$$

The axial load, call it "P" will create a state of *normal stress*, σ, in the material, defined as the load divided by the cross sectional area, "A" of the part. In the case of a simple bar in tension then, the normal stress is:

$$\sigma = P/A$$

Axial stress and strain are linearly related to one another and as long as the material is not stressed beyond a certain point, called the yield strength, it will behave in a linearly elastic manner under loading. That means it stretches under the load, but if the load is removed, it returns to its original length with no permanent, or plastic, deformation.

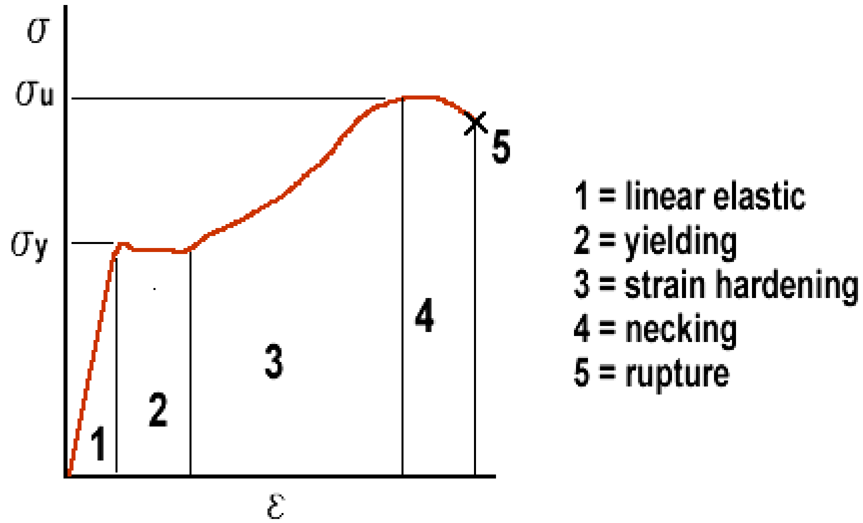
The stress and strain are related to one another by *Hooke's Law*, given by:

$$\sigma = E\varepsilon$$

Where E is the *Young's Modulus*, a material constant specific to each material. In a sense, the Young's Modulus defines how inherently stiff the material is. Most steels, for example, have a Young's Modulus of around $3x10^7$ psi, while most aluminum alloys have a Young's Modulus of around $1x10^7$ psi. This means that under a given load, if you have two identical parts, one made from steel, and the other from aluminum, the aluminum part will deform or stretch three times as much as the steel part.

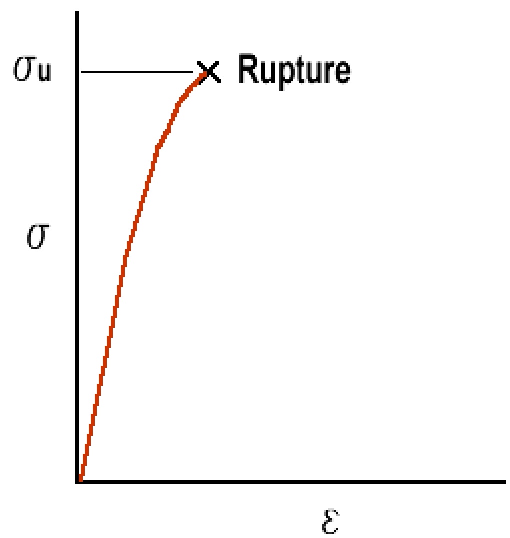
One of the most fundamental tests of the static strength of materials is the *tensile test*. Many times, the results of the static test are all the designer will have to go on to make decisions about the strength of the designed part. The object of the test is to obtain the stress-strain diagram for the part. In this test, a specimen is subjected to increasing axial load, and the deformation is measured precisely and plotted against the applied load/area or applied stress.

When a ductile material undergoes a tensile test, the stress-strain diagram looks something like this (although the shape varies for different materials):



Typical Stress-Strain Curve for Low Carbon Steel

The yield strength of the material is shown as σ_y and the ultimate strength is denoted σ_u. Units for the yield and ultimate strength of the material are generally given in psi, ksi, or MPa. (1 MPa = .14503 ksi = 145.03 psi). Hooke's Law applies in region 1, while the material is experiencing elastic deformation.



Typical Stress-Strain Curve for Cast Iron (Brittle)

A **ductile material** will experience a significant amount of plastic deformation prior to failure. One of the quantities measured in a tensile test is the *percent elongation* which is the difference between the original length and the length at rupture, divided by the original length, x 100. *Materials with a percent elongation of 5% or more are generally considered ductile, and those with 5% or less are considered brittle.*

The second curve is for a tensile test of a brittle material. Note the absence of significant yielding (non-linear part of curve) prior to failure.

Another measure of ductility related to percent elongation is *percent reduction in area.* This is a measure of how much necking occurs in a tensile specimen prior to final failure. The higher the number, the more ductile the material is considered to be.

Strength of the material is thus usually specified as tensile yield strength and ultimate tensile strength. Other strength quantities include compressive strength, shear strength and torsional yield strength. For many materials compressive strength is nearly equal to tensile strength, but for some, such as brittle materials like cast iron, there may be a large difference. Brittle materials tend to be stronger in compression than in tension, and consequently the compressive strength must usually be considered separately for brittle materials if compressive loading is a possibility.

Hardness of the material is another of the basic properties. Most hardness testing involves the application of a concentrated compressive load on an indenting probe. The size and/or depth of the indentation is measured, and converted to the appropriate hardness scale. (Hardness scale conversions are easily found online). In general for most metals, harder = stronger in terms of tensile strength. An increase in hardness, however, is usually obtained at the expense of ductility. Thus, harder materials may be stronger from a tensile strength perspective, but for many applications, ductility is a highly desirable property. One of the reasons ductility is desired is it represents the ability of a material to absorb energy in the form of plastic strain.

Impact strength relates to how much energy a material can absorb as it fractures. Most impact strength tests are done by swinging a weighted pendulum to strike a notched bar of material, and measuring the height of swing of the pendulum post-strike. The change in height of swing is kinetic energy absorbed by the specimen during the process of fracture. Two popular tests are called the *Charpy impact test* and the *Izod impact test*. Results are usually reported in ft-lb of energy absorbed. The higher the ft-lb, the higher the impact strength. Impact strength is highly dependent on temperature, and this must be part of the overall design considerations when impact loading is possible.

3.0 Statistical Variations

Virtually all materials are subject to variations in chemistry, processing variations, impurities, etc., all of which lead to variations in material properties like tensile strength and yield strength. The designer must acknowledge and deal with these variations. Other factors subject to statistical variations could include loading, tolerances on machined or extruded parts, thicknesses of coatings, etc. It is up to the designer to consider statistical variation as required, as part of his or her design methodology.

The important concept to understand is that for a given value, say, tensile yield strength, the values will vary, and they will probably (although not always) vary according to a normal distribution. There are relatively straightforward ways to determine if a set of data from a sample follows a normal distribution, and how well it fits. (For more on this, consult a statistics text). Assuming your data fits the normal distribution, what you are really interested in is how much variability there is, and what conclusions can be made relevant to applicability of stated values (such as tensile yield strength) to your design situation.

Statistical analysis usually involves taking a sample of values from a population, and extending the results of the sample to the overall population.

First a few definitions:

population mean: $$\mu = \frac{1}{N}\sum_{1}^{N} x_j$$

sample mean: $$\bar{x} = \frac{1}{N}\sum_{1}^{N} x_j$$

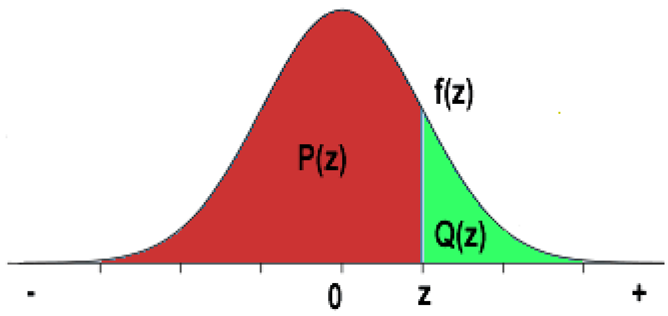
sample variance: $$s_x^2 = \frac{\sum x^2}{N} - \bar{x}^2$$

sample standard deviation: $$s_x = \left[\frac{\sum x^2 - \frac{\left(\sum x\right)^2}{N}}{N-1} \right]^{1/2}$$

The mean, standard deviation and variance are all easily calculated with most scientific calculators.

The Normal (or Gaussian) Distribution

The variations in many populations (sets of data) will be found to fit a particular type of probability distribution called a *normal distribution*. The normal distribution is the familiar bell-shaped curve which represents the frequency and variation of data points from the mean.



Standard Normal Distribution
P(z) = 1- Q(z)

The total area under the curve which describes the standard unit normal distribution, from x = -∞ to x = +∞, equals one square unit. The area between any two points represents the relative proportion of the distribution that lies between those two points.

The value "z" is defined as the *standardized variable*, and is given by:

$$z = \frac{x - \mu}{s_x}$$

Through the standardized variable, the distribution of any normally distributed population data may be related to the unit normal distribution. The equation for the unit normal distribution using the standardized variable z is:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$$

For any value of z, the area under the curve can be determined by numerical methods, the Excel function "normdist()", scientific calculator, or by looking up the value in standard statistical tables. The area found represents the proportion of the data which will lie above or below the value specified.

Example: A machined lot of 500 brackets is sampled, and the mean of the load capacity is found to be 5000 lb, with a standard deviation of 250 lb. The acceptable minimum load capacity is 4500 lb. How many brackets will fail this criteria? (Assume a normal distribution).

Solution: First, calculate the standardized variable, z:

$$z = \frac{x - \mu}{s_x} \quad = (4500\text{-}5000)/250 = -2$$

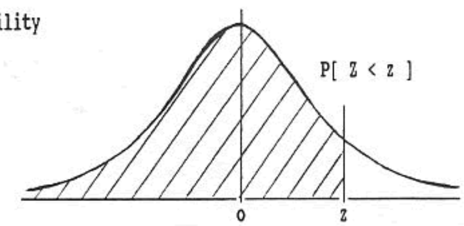
Since the normal distribution is symmetrical, we can use the positive value for z. From a standard table for z = 2.0, we find P(z) = 0.9773.

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z i.e.

$$P[Z < z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\, dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

Since P(z)+Q(z) = 1, we know that Q(z), the area of the left side "tail" is:

$$Q(z) = 1 - 0.9773 = 0.0227$$

So this is the proportion of area under the curve to the left of -z.

Then, the total number of brackets with a load capacity of less than 4500 lb will be:

$$0.0227 \times 500 = 11.375$$

Thus, about eleven or twelve brackets will be expected to fail the criteria.

4.0 Heat Treatment

Metals are often subjected to various heat treatment processes to obtain properties which are desired for a particular design, such as strength, toughness, hardness, ductility, fatigue resistance, etc. Heat treatment usually involves a number of specific steps involving temperature and time, and which cause changes to the crystalline structure of the metal being processed. Some of the more common processes are:

Annealing: Slow cooling of hot material. It may be done for a number of reasons including softening, removal of residual stresses, purification, increasing ductility, or to produce certain changes in crystalline structure.

Hardening: Heating and then quenching, usually of steels. Increased hardness is generally accompanied by increased strength.

Normalizing: Heating to slightly above the critical temperature (steels) and slowly cooling in air. Generally used to refine grain structure, remove residual stresses, and improve toughness.

Quenching: Rapid cooling of hot material by immersion in cool liquid or gas. associated with formation of martensite.

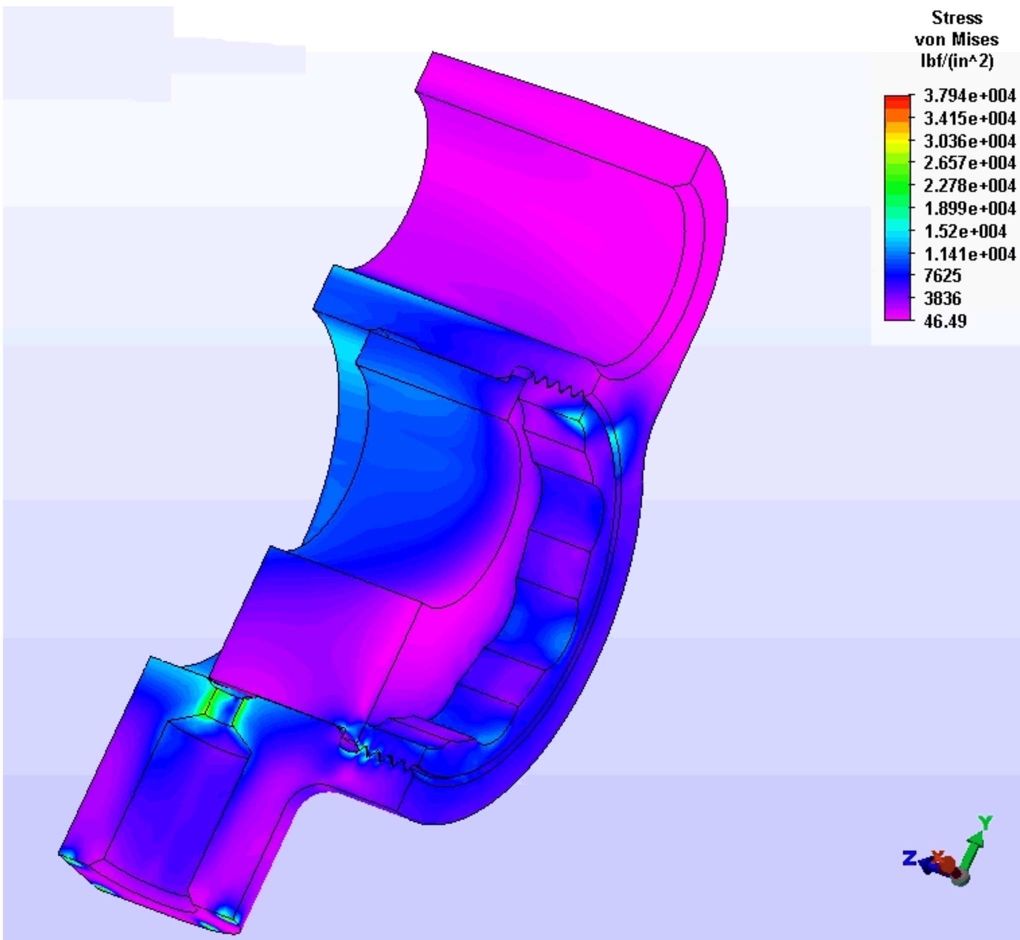
Spheroidizing: Application of certain sequences of heat and cool to produce spheroid carbides in steels. Generally used to improve machinability of steels.

Tempering: Reheating to a temperature below the critical temperature, followed by a specific cooling rate. Generally used to achieve a certain properties like hardness, strength and ductility. In aluminum alloys, designated by the suffix "T".

5.0 Stress Concentrations

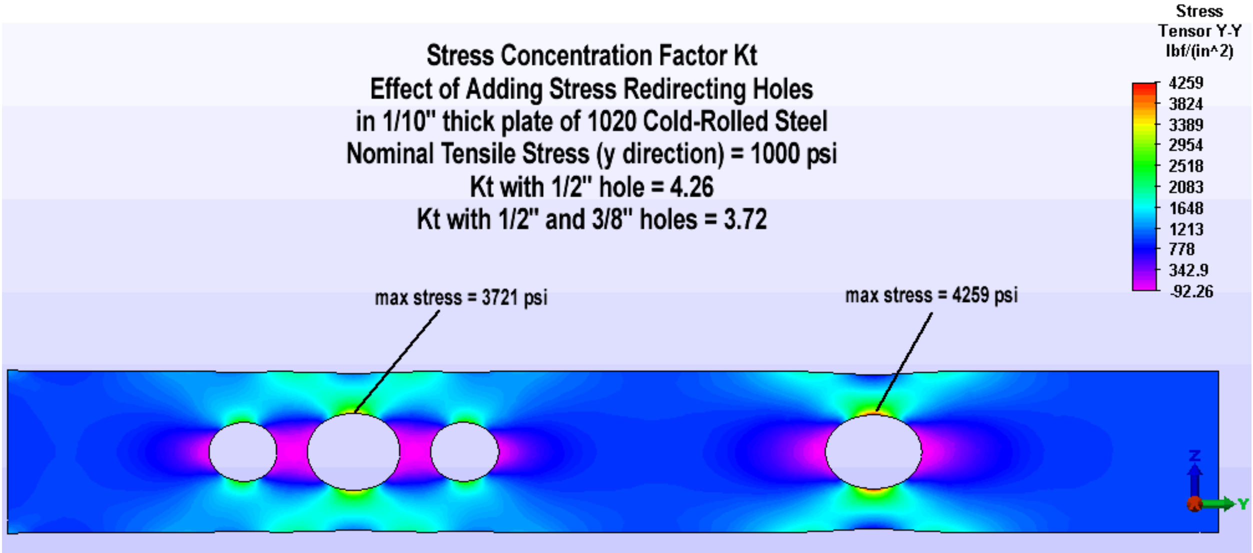
A stress concentration is any discontinuity in a part which elevates the stresses in the region of the discontinuity. Examples include cross-holes, grooves, changes in section, slots, threads, bosses, flanges, etc.

Here is an example from a finite element analysis (FEA) assembly contact model of a small high pressure tool cylinder/housing with a cross-hole (oil port) and a threaded rear end cap. The cross-hole is a stress concentration, as well as the thread in the end of the cylinder.



For many common changes of section or features such as cross-holes, slots, etc., there are standard charts available for determining stress concentration. Prior to the widespread availability of FEA, stress concentrations were determined theoretically or experimentally by techniques such as strain gauging, brittle coating and photoelastic techniques.

Stress concentrations can be though of as bottlenecks for field lines of stress. Considering a plate with a hole in tension, imagine the stress field as straight lines running through the part. The hole causes these lines to deflect, elevating the stress field in the region of the hole. In fact, this analogy to field lines deflecting is not just a good visualization tool. It can be used on real parts to reduce the stress concentration by "guiding" the stress field lines around the discontinuity at a gentler rate of change. A part can actually be made stronger sometimes by *removing* material. Here is an example from an FEA model. The stress concentration, Kt, is less where the added smaller holes have been placed to redirect the stresses around the larger hole.



The stress concentration factor for normal stress, K_t, is defined as the ratio of the actual maximum stress in the region of the discontinuity to the nominal stress through net section or gross section of the part. Pay particular attention to whether *net section* or *gross section* is referenced when using stress concentration charts. *Net section* is the reduced section in the area of the discontinuity (in the case above, it would be the width of the bar minus the hole diameter times the thickness). *Gross section* would be the width of the bar without the hole times thickness.

$$K_t = \frac{\sigma_{\max}}{\sigma_0}$$

For shear stress concentration factor, use K_{ts} instead of K_t.

$$K_{ts} = \frac{\tau_{\max}}{\tau_0}$$

Application of stress concentration factors in design depends to a high degree on the ductility of the material being evaluated. Since stress concentration is a localized effect, in a ductile material subjected to static loading, the material will yield plastically in the area of the stress concentration when it is first loaded. For this reason, as long as the material is ductile and the loading is static, it is not necessary to apply the stress concentration factor when computing stresses. For brittle materials, however, the stress concentration factor should be used even in static design scenarios because a brittle material will not yield, and thus there will be no plastic relief of stresses around the stress concentration. So for brittle materials and a single stress concentration, simply multiply the nominal stress by the appropriate stress concentration factor - but first adjust the applied stress concentration factor to account for the effects of notch sensitivity.

We can define the fatigue stress concentration factor (which is also used as an effective stress concentration factor for ductile materials), K_f as

$$K_f = 1 + q(K_t - 1)$$

Where q is the notch sensitivity factor, a material constant. For example, for most cast iron materials the notch sensitivity is low, from 0 to 0.2. We can typically use 0.2 for cast irons in general and be designing conservatively. For brittle materials with stress concentrations then, we can look up the notch sensitivity factor, q, and calculate K_f, then use K_f to either reduce the allowable stress, or increase the applied stress. From a design standpoint it makes no difference which approach is used, and in fact both approaches are commonly used.

Other cases where K_t (or K_f) should be applied in static loading are for highly cold-worked materials, very high strength materials, and case hardened materials.

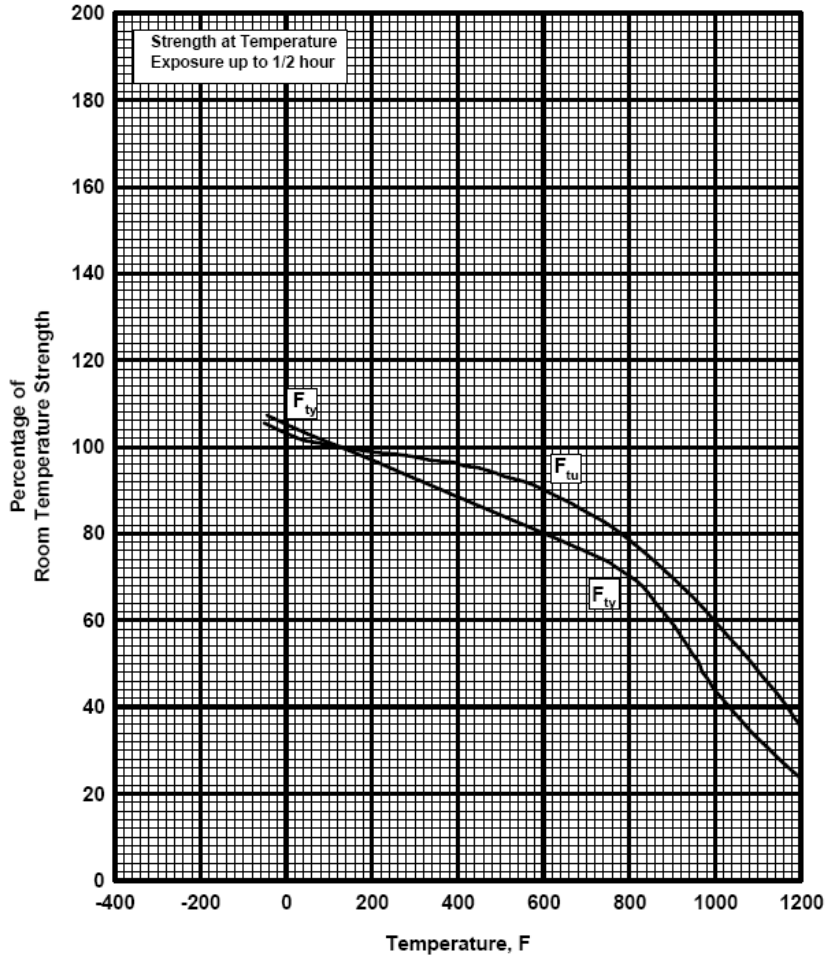
It should be noted also that if a part will be subject to repeated (fatigue) loading, than for both ductile and brittle materials a fatigue stress concentration factor K_f is used. (This course will not address fatigue loading).

6.0 Temperature

Elevated or reduced temperatures will significantly affect the material properties of all engineering materials. If elevated or reduced temperatures are part of the design parameters, they must be considered and accounted for. The best method of design verification is actual testing of actual parts in the environmental conditions they will be subjected to.

Reduced temperatures, in addition to changing material properties in general, may also cause a transition from ductile to brittle behavior in otherwise ductile materials. The temperature at which this transition takes place is called the *nil ductility transition temperature (NDT or NDTT)*. Essentially, the impact strength and elongation of a material undergoes rapid decreases in the region of the nil ductility temperature. Design of critical parts and structures must take this into account and provide a margin of temperature difference between the NDT and the minimum temperature the structure or part will be subjected to in service. (If designing critical parts or structures, be sure to comply with the appropriate codes and standards which govern the design of such parts.)

Elevated temperatures in general reduce the strength of metals, and long term exposure to elevated temperatures can cause permanent deformations, even when stresses remain below the yield strength of the material. This phenomenon is known as creep, and is a significant factor when designing with non-linear materials such as plastics, or parts subjected to long-term significant elevated temperatures. In essence, elevated temperatures can cause materials which behave in a linear elastic way at room temperature, to behave like a non-linear plastic. It is highly recommended that creep properties be well understood for the particular material and design conditions and verified by long-term testing. The following figure shows the effects of elevated temperature on tensile and yield strength of AISI low alloy steels.



Effect of temperature on the tensile ultimate strength (F_{tu}) and tensile yield strength (F_{ty}) of AISI low-alloy steels (all products).

(from MIL-HDBK-5H, 1998)

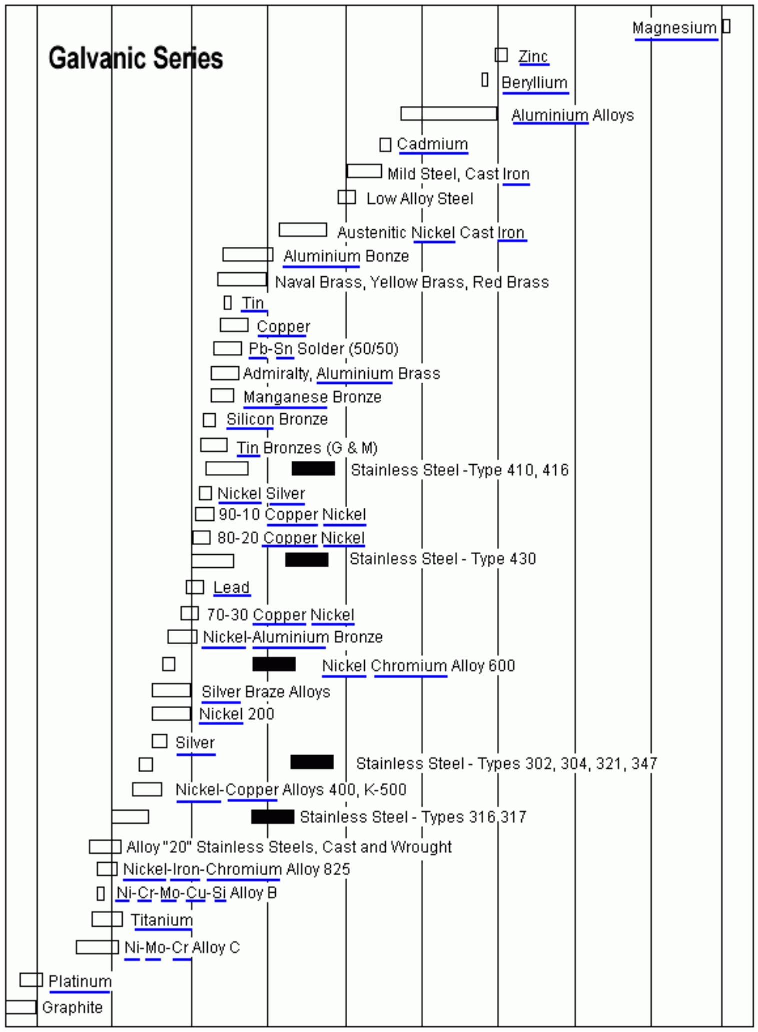
7.0 Corrosion

Corrosion is the deterioration of the properties of a material as a result of interacting with the environment to which it is exposed. Three basic type of corrosion which may affect parts from a static strength point of view are simple environmental corrosion, galvanic corrosion, and stress corrosion.

Environmental corrosion is caused by chemical interaction with liquids, solids, or gases in the environment. Corrosion is a significant factor in the design of any structures, machines or vehicles which will subjected to outdoor elements, elevated temperatures, corrosive chemicals, marine environments, radiation, ozone, etc. Environmental corrosion can be mitigated by selecting appropriate materials resistant to corrosion, by surface coating, or surface maintenance. One of the most commonly used tests to determine resistance to corrosion is a salt-spray test, of which there are many variants.

Galvanic corrosion is caused by a flow of current between dissimilar metals in electrical contact. It causes surface damage in the form of uniform corrosion and/or chemical attack. Anywhere two dissimilar metals are in contact, galvanic corrosion must be considered a possibility.

Galvanic corrosion can be mitigated by application of non-conducting coatings between parts, or by sealing openings in cracks between parts (thus preventing electrolytic compounds from entering). Other methods include increasing the size/surface area of the anodic member (more actively corroding member) relative to the cathodic (more inert) member, cleaning, or choosing metals which are close in the galvanic series.



(from: www.corrosionsource.com)

Stress corrosion or *stress corrosion cracking* is the phenomenon of failure of normally ductile materials under the combined influences of tensile stress and a corrosive environment. Under the influence of tensile stresses, be they applied stresses due to load or residual stresses due to processing, materials may corrode and crack, that ordinarily would not be considered susceptible to corrosion. The mechanism of stress corrosion cracking is similar the that causing galvanic corrosion. Stress levels of 50-75% of yield strength are usually required for stress corrosion cracking to take place, but in some cases lower stresses can still cause it.

Stress corrosion cracking can be mitigated by heat treatment to reduce or eliminate the residual stresses from processing. Shot peening is another, preferred method for removing surface residual tensile stresses and replacing them with a protective layer of residual compressive stress. Design changes which prevent the development of residual or assembly stresses (such as reducing the interference of a press-fit, or reducing installation torque on a tapered plug) may be a simple fix as well, as long as function is not adversely affected. Protective coatings to prevent corrosive agents coming in contact with exposed stressed surfaces may also be used.

8.0 Safety Factors and Allowable Stress

The factor of safety in a static design, in terms of design stress or allowable stress, σ_a, and material strength, S is:

$$n = S/\sigma_a$$

For example, if we have a part where tensile yield strength is the governing factor, with a tensile yield strength Sy = 90,000 psi, and our allowable stress is 30,000 psi, the n = 90,000/30,000 = 3.0.

Accounting for uncertainty in the design material strength and load variability may be achieved by using a combined safety factor, n_c.

$$n_c = n_s n_l$$

Where n_s is a factor which relates to uncertainty in the strength of the material or part and n_l is a factor which relates to uncertainty in the loading of the part. If, for example you know the yield strength of the material is 100,000 psi with a standard deviation of 5000 psi, and you want 99% reliability relevant to material strength, then you could use statistics to determine a factor n_s which would account for material variation (assuming a normal distribution applies). We can either calculate or use tables to find the z value which corresponds to 99% of the area under the curve (thus leaving 1% for the "tail"). From the chart on page 6, we can interpolate to find z = 2.326 for a P(z) of 99%. We can then compute the reliability factor, K_r from the following formula:

$$K_r = 1 - \frac{s_x}{\mu} z$$

$$K_r = 1 - \frac{5{,}000}{100{,}000} 2.326 = 0.8837$$

And then:

$$n_s = 1/K_r = 1/0.8837 = 1.132$$

as the factor to account for the yield strength variation.

If you want to also account for a variation in loading of +30 percent, for example, you could apply a load variability factor of 1.3, in combination with the material factor of 1.132 to get a total factor:

$$n_c = n_s n_l = (1.132)(1.3) = 1.472$$

Thus, the allowable tensile stress using the combined safety factor would follow from:

$$n_c = S / \sigma_a$$

then $\sigma_a = S / n_c = 100{,}000 / 1.472 = 67{,}930\, psi$

It then may be further desired to apply an overall safety factor once load variability and material variability have been accounted for, depending on the nature of the application.

It is up to the designer to consider every possible angle when applying safety factors. Some of the issues to consider include, but are not limited to:

1) Consequences of failure with respect to safety and functionality

2) Governing codes

3) Sources of load variability (impact, human error, unexpected overload, machine vibration, etc.)

4) Environmental influences (temperature, corrosion, weather related issues, etc.)

5) Part geometric variability, tolerances

6) Accidental damage or misuse/abuse

7) Degree of verification testing

8) Material variability

9) Functional consequences of over design (for example, most airplane structural parts could not possibly carry a 4:1 static factor of safety - the plane would be too heavy to fly - but in the hydraulic power tool industry, a 4:1 static factor of safety is fairly common practice)

10) Ductile or brittle materials

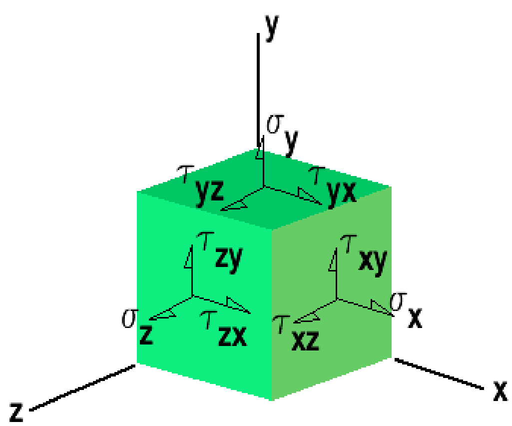
11) Can failure be anticipated, observed, avoided, inspected, monitored

12) Cost of replacement

13) Collateral damage to dependent parts or assemblies

9.0 Combined Stress Failure Theories - General

If the state of stress at the critical location(s) involves a single stress in one direction, (a *uniaxial* stress), then it is fairly simple to relate the tensile test properties to the stress and determine if a part will fail or not. When, as is usually the case, stress is *multiaxial*, (in the absence of or prior to actual product testing), it becomes necessary to relate the results of properties obtained by uniaxial tests to the multiaxial stress condition and make a decision about the integrity of the part. There are a multitude of theories to this effect, called *combined stress theories*. Not all will be presented here - just the ones commonly used. Over the course of time, tests have shown that certain failure theories fit the experimental data better than others depending on the properties of the materials being analyzed. Selection of which failure theory to use depends largely on whether the material is ductile or brittle. In the case of ductile materials, it also depends on whether or not there is a significant difference between the tensile yield strength, σ_{yt}, and the compressive yield strength, σ_{yc}. In the case of brittle materials, it similarly depends on whether or not there is a significant difference between the ultimate tensile strength, σ_{ut} and the ultimate compressive strength, σ_{uc}. Of course, if you are governed by a specific design code, you must follow the appropriate code, even if it conflicts with the table presented here. In general, remember, ductile materials have elongation > 5% in 2", and brittle materials have elongation <5% in 2". Also note that in any of the following theories which use principal stresses, they are arranged such that:



Multiaxial Stress State

$$\sigma 1 \succ \sigma 2 \succ \sigma 3$$

Selection of Appropriate Static Failure Theory - Ductile Isotropic Materials	
Tensile Yield Strength (σ_{yt}) vs. Compressive Yield Strength (σ_{yc}) Relationship	Appropriate Failure Theory
σ_{yt} approximately equal to σ_{yc}	Distortion Energy Theory (a.k.a. Von Mises - Hencky Theory, - equivalent to Octahedral Shearing Stress Theory)
	Maximum Shear Stress Theory (a.k.a. Tresca-Guest Theory)
σ_{yc} significantly different than σ_{yt}	Mohr's Theory

Selection of Appropriate Static Failure Theory - Brittle Isotropic Materials	
Ultimate Tensile Strength (σ_{ut}) vs. Ultimate Compressive Strength (σ_{uc}) Relationship	Appropriate Failure Theory
σ_{ut} approximately equal to σ_{uc}	Maximum Normal Stress Theory
σ_{uc} significantly different than σ_{ut}	Modified Mohr's Theory

10.0 Distortion Energy (Von Mises - Hencky, or Octahedral Shearing Stress) Theory

This theory uses the principal stresses, σ_1, σ_2, σ_3; and the uniaxial failure strength σ_f. The formula is based on the theory that failure will occur if the distortion energy per unit volume which is the result of a multiaxial state of stress equals or exceeds the distortion energy per unit volume at the instant of failure in a uniaxial stress test (such as the tensile test). Thus failure is predicted to occur if the following expression is true:

$$\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]\geq \sigma_f{}^2$$

for the common situation of bi-axial stress the von Mises based failure criteria reduces to (fails if true):

$$\sigma_1{}^2-\sigma_1\sigma_3+\sigma_3{}^2\geq \sigma_f{}^2$$

And for the special case of bending and torsion, failure occurs of the following is true:

$$\sigma_x^{\ 2} + 3\tau_{xy}^{\ 2} \geq \sigma_f^{\ 2}$$

11.0 Maximum Shear Stress (Tresca-Guest) Theory

While the distortion energy theory is preferred, the maximum shear stress theory is easy to apply, is conservative, and is used in many design codes. Maximum shear stress theory is only used to predict onset of yielding. The maximum shear stress theory states that failure will occur if any of the following expressions are true:

In principal shear terms: $|\tau_1| \geq |\tau_f|$ **or** $|\tau_1| \geq |\tau_f|$ **or** $|\tau_1| \geq |\tau_f|$

Where: $\tau_f = \dfrac{\sigma_f}{2}$ = the failure value for the principal shear stress corresponding to the failure value for the principle normal stress from the uniaxial test.

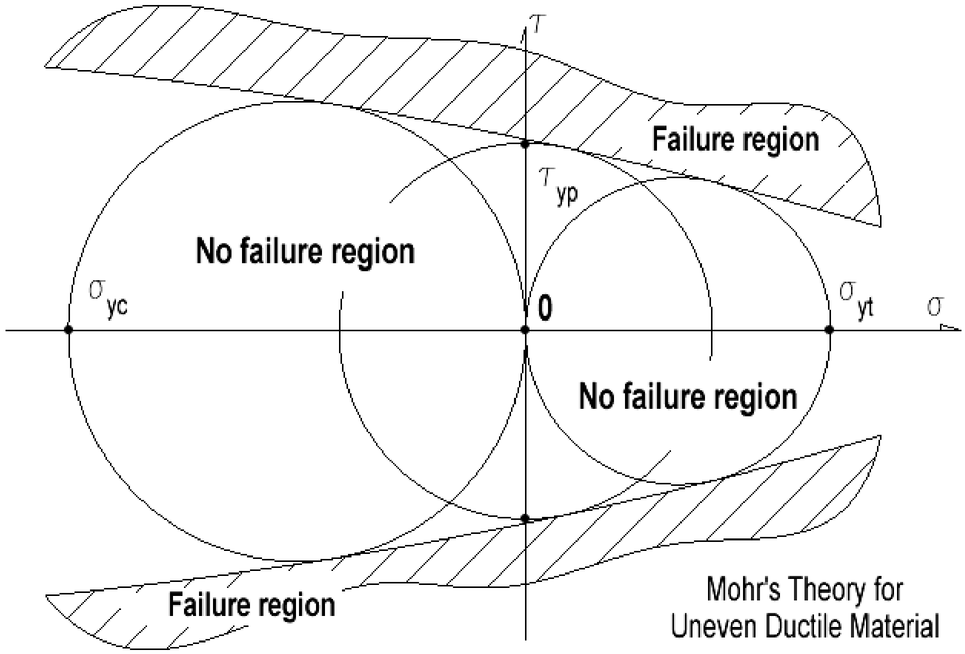
And the equivalent expression in principal stress terms states that failure will occur if any of the following are true:

$|\sigma_1 - \sigma_2| \geq |\sigma_f|$ **or** $|\sigma_2 - \sigma_3| \geq |\sigma_f|$ **or** $|\sigma_3 - \sigma_1| \geq |\sigma_f|$

The maximum shear stress theory states that the yield strength in shear is one-half of the tensile yield strength.

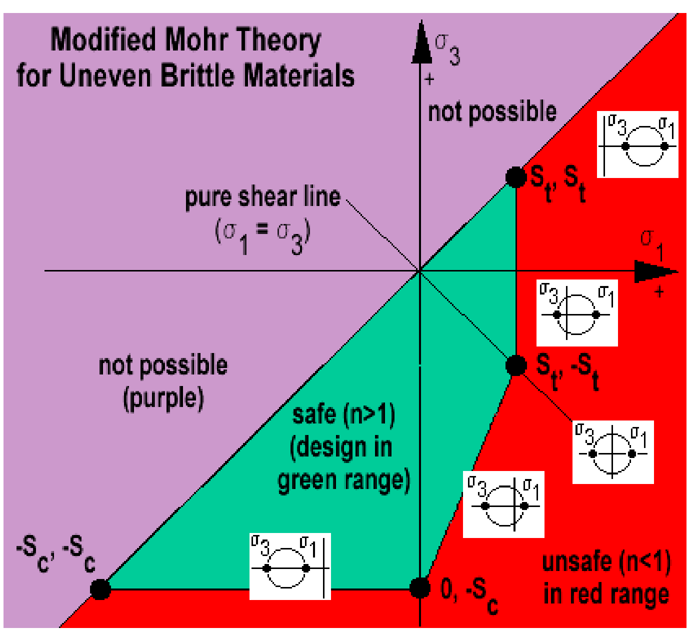
12.0 Mohr's Theory

For the unusual case of ductile materials with significantly different values of tensile and compressive yield strengths (uneven ductile materials), the graphical method of *Mohr's Theory* may be used. In this method, the results of uniaxial tensile test(s), compressive test(s), and if available, torsional shear test(s) can be graphed, constructing circles for each test on the $\tau - \sigma$ plane in a manner similar to that used to construct the familiar Mohr's circle.



The best curve tangent to the Mohr's circles from the uniaxial tests is constructed. Then, failure is predicted to occur in a multiaxial state when the largest conventional Mohr's circle associated with the state of stress in the part becomes tangent to or exceeds the boundaries defined by the tangent curve to the Mohr's circles created from the uniaxial test specimen data.

13.0 Modified Mohr's Theory



The *modified Mohr's theory* fits experimental results well and is used for brittle materials where there is a significant difference between the ultimate compressive strength and the ultimate tensile strength (uneven brittle materials). It is best explained graphically.

The failure criteria vary by where the stress point (determined from the maximum and minimum principal stresses) lies according to the graph shown. If the state of stress lies inside the green zones it is in the "safe" zone and the factor of safety n >1. If it is in the red zones it is in the failure region and n < 1. This type of problem can be solved graphically by constructing the graph and then determining if the particular state of stress at the critical point(s) falls in the green range. The border between green and red represents n=1. It may also be solved numerically according to the following formula:

$$1/n = \text{maximum of} : (\sigma_1/S_t, \; -\sigma_3/S_c \,, \; \sigma_1/S_t - (\sigma_1 + \sigma_3)/S_c)$$

14.0 Maximum Normal Stress Theory

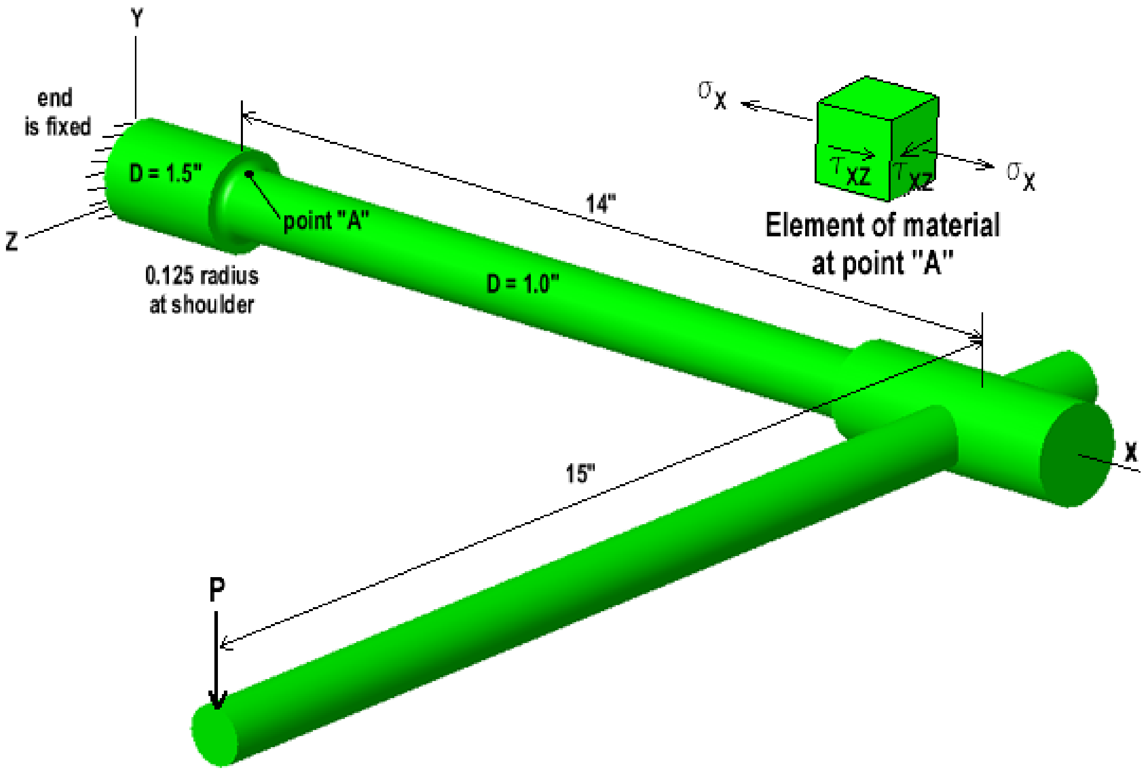
This theory uses the principal stresses, σ_1, σ_2, σ_3; the tensile failure strength σ_t, and the compressive failure strength σ_c. Failure is predicted to occur if **either** of the following expressions are true:

$$\sigma_1 \text{ or } \sigma_2 \text{ or } \sigma_3 \geqq \sigma_t \qquad\qquad \sigma_1 \text{ or } \sigma_2 \text{ or } \sigma_3 \leqq \sigma_c$$

The maximum normal stress theory should only be used for even brittle materials. It is overly conservative for ductile materials, and can be unsafe for uneven materials in general.

15.0 Combined Failure Theory Examples

15.1 Example: Application of Combined Stress Failure Theories - Ductile Materials



For a torsion bar, the loading is as shown. At what load "P" will the part yield at point "A" if the bar is made from a steel with 18% elongation and tensile and compressive yield strength = 81 ksi?

Since this is a ductile part, and no design codes are telling us to do otherwise, let's use the von Mises (distortion energy) theory. *Also, since the material is ductile and this is a static loading case, we will not apply any stress concentration factors.* This is also a special case of bending and torsion, so we can bypass Mohr's circle and use the formula:

$$\sigma_x{}^2 + 3\tau_{xy}{}^2 \geq \sigma_f{}^2$$

So we need to find the shear stress:

$$\tau_{xy} = \frac{Tr}{J}$$

Where T = 15P, r = 0.5 and J = the polar moment of the 1" bar, which is given by :

$J = \frac{1}{2}\pi r^4$ substitution gives $\tau_{xy} = \frac{30P}{\pi r^3}$

For the bending stress, σ_x we have:

$$\sigma_x = \frac{Mc}{I}$$

where c = distance from the neutral axis, M is the bending moment (in this case M = 14P) and I = the moment of inertia of the bar, which is given by:

$$I = \frac{1}{4}\pi r^4.$$

Then, substitution gives:

$$\sigma_x = \frac{56P}{\pi r^3}$$

So we can now write the failure criteria in terms of the load P, as:

$$\left(\frac{56P}{\pi r^3}\right)^2 + 3\left(\frac{30P}{\pi r^3}\right)^2 \geq \sigma_f^{\ 2}$$

Rearranging, we get:

$$P \geq \sqrt{\frac{\pi^2 r^6 \sigma_f^{\ 2}}{5836}}$$

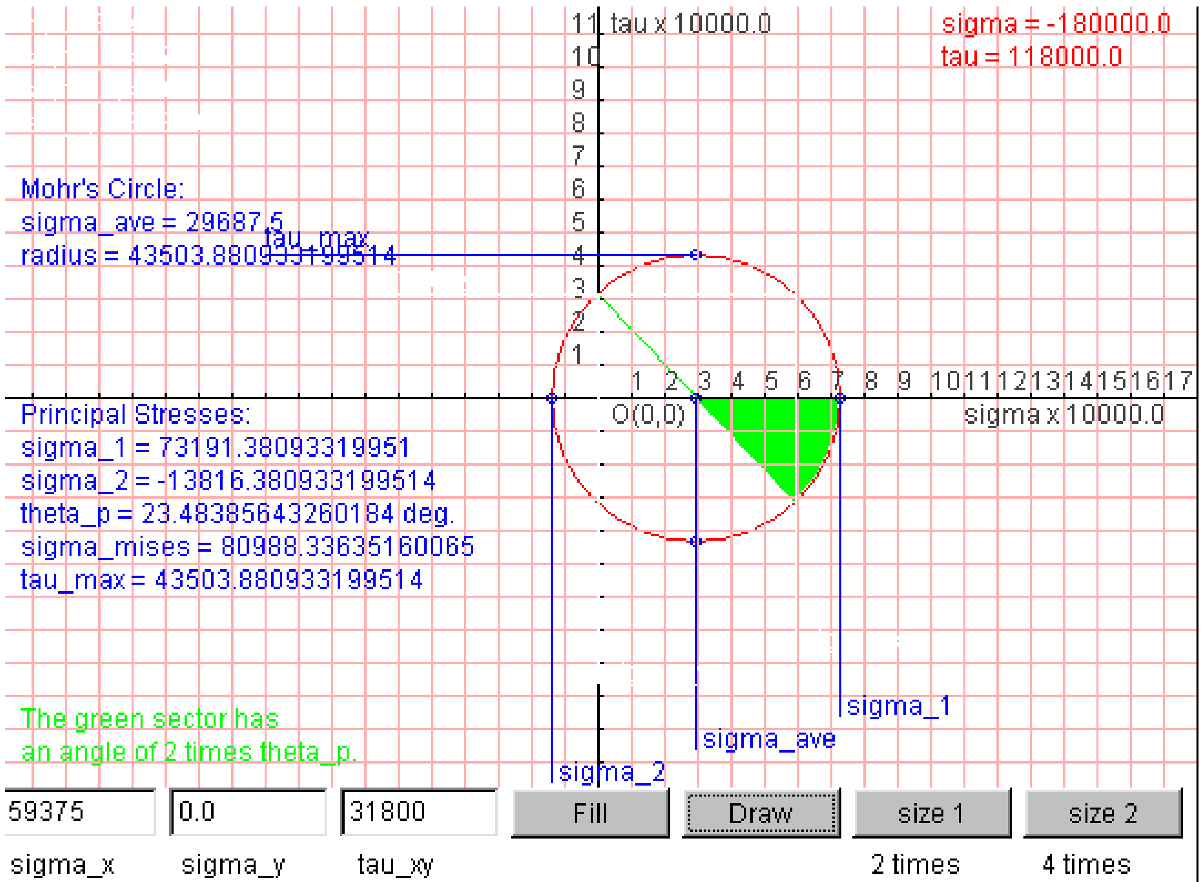
Substituting in 0.5 for r, and 81 ksi for the failure strength, we get:

P = 416.4 lbf for the load which will yield the bar.

There are many online resources available for calculating Mohr's circle principle stress and von Mises stress, like the one here:

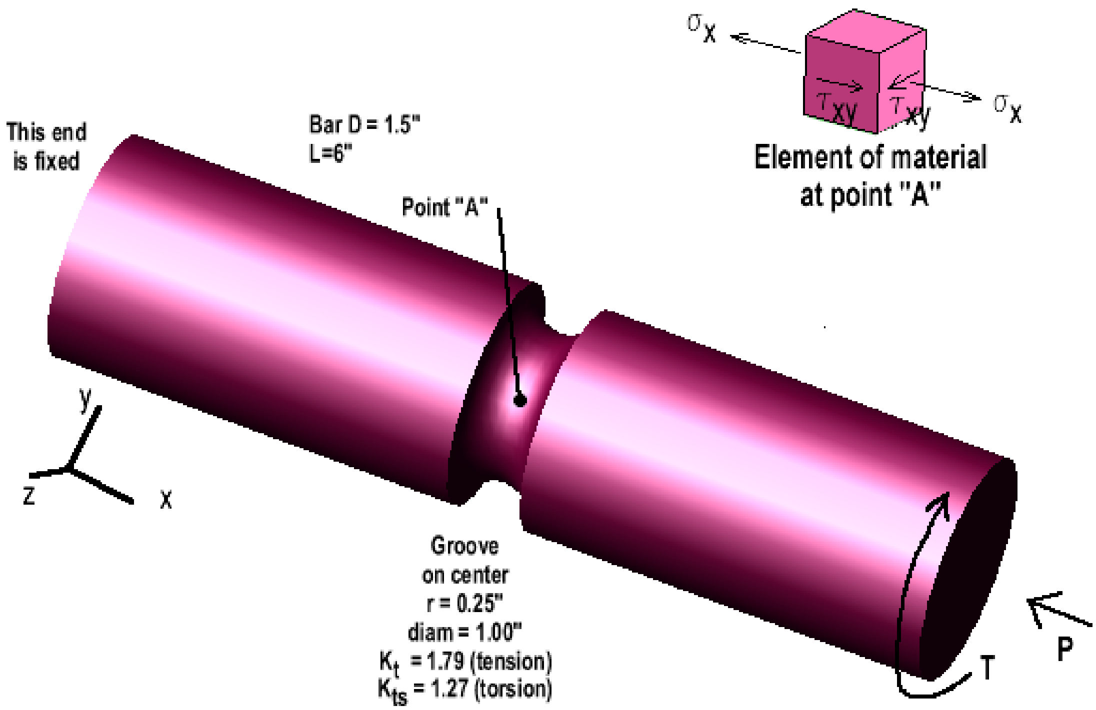
http://www.aoe.vt.edu/~jing/MohrCircle.html

You provide the normal stresses and shear stress, and it provides the principle stresses and von Mises stress. The Mohr's circle for the above problem was run and the results look like this:



15.2 Example: Application of Combined Stress Failure Theories - Brittle Materials

For a pin with a full radius groove, the loading is as shown. If P = 20,000 lbf and T = 2000 in-lb, what is the safety factor? Assume no material or load variability. The pin is made from an ASTM 40 cast iron with tensile strength = 42 ksi and compressive strength of 140 ksi. Notch sensitivity, q = 0.2.



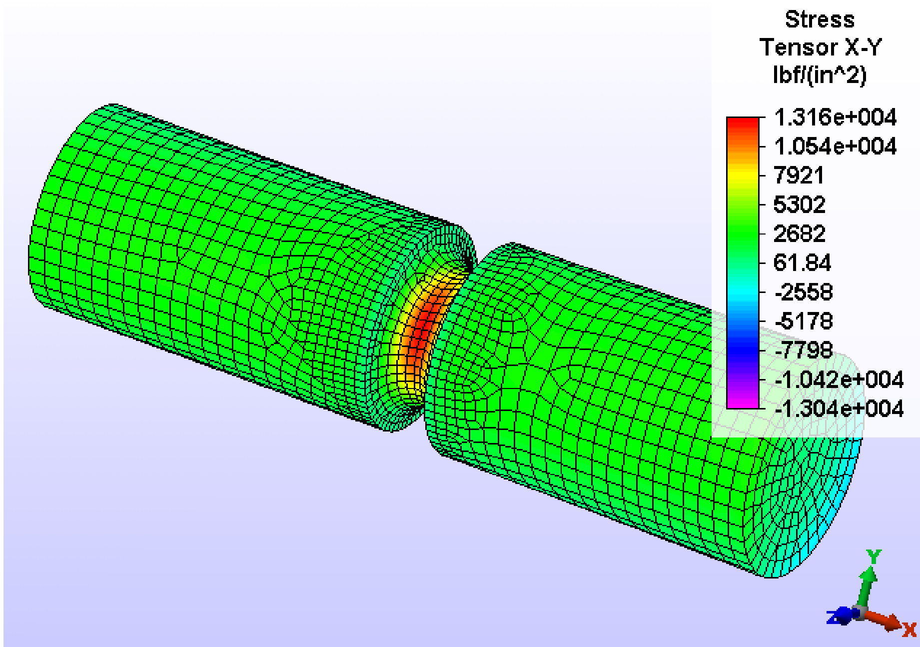
Since this is an uneven brittle part, and no design codes are telling us to do otherwise, let's use the modified Mohr's theory. *Also, since the material is brittle we will apply the stress concentration factors, adjusted for the notch sensitivity.* For the loading shown at point "A" we will have a shear stress due to torsion from T, and A compressive normal stress due to the axial load P. Looking at the torsion, we know for a round bar that

$$\tau_{xy}(\max) = \frac{Tr}{J} \; ; \quad \text{and we know} \quad J = \frac{1}{2}\pi r^4$$

$$\text{so} \quad \tau_{xy} = \frac{2T}{\pi r^3} = \frac{2(2000)(in-lb)}{\pi (0.5)^3 (in^3)} = 10{,}190(psi)$$

$$\text{and} \quad K_{ts} = 1.27 \quad \text{so} \quad \tau_{xy}(corrected) = (1.27)(10{,}180) = 12{,}940\, psi$$

And our FEA model verifies this…



We do not, however want to use the full value of the Kt calculated either from the FEA or from stress concentration charts due to the low notch sensitivity of the material. We adjust the Kt using the formula for notch sensitivity:

$$K_f = 1 + q(K_t - 1) = 1 + (0.2)(1.27 - 1) = 1.054$$

so our actual shear stress for design purposes is:

$$\tau_{xy}(corrected) = (1.054)(10{,}190) = 10{,}740\, psi\ .$$

Now to find the corrected axial compressive stress. First we can calculate a nominal compressive stress in the net section simply by dividing force by area:

$$\sigma_x = \frac{(-20{,}000)(lbf)}{\pi 0.5^2 (in^2)} = -25{,}460\, psi$$

And we compute K_f from the given K_t (for axial loading) and the notch sensitivity:

$$K_f = 1 + q(K_t - 1) = 1 + (0.2)(1.79 - 1) = 1.158$$

Then, $\sigma_x(corrected) = (1.158)(-25{,}460) = -29{,}480\, psi$.

We construct a Mohr's circle, either manually or with a program and get the maximum and minimum principle stresses for use in the modified Mohr's theory.

This gives is σ_1 = 3500 psi and σ_3 = -32,980 psi. From the modified Mohr's theory we have:

$$1/n = \text{maximum of : } (\sigma_1/S_t, \;\; -\sigma_3/S_c , \;\; \sigma_1/S_t - (\sigma_1 + \sigma_3)/S_c)$$

σ_1/S_t = 3500/42000 = .0833 ;

$-\sigma_3/S_c$ = -(-29,385)/140000 = 0.236

$\sigma_1/S_t - (\sigma_1 + \sigma_3)/S_c$ = 0.0833 - (-.211) = 0.294

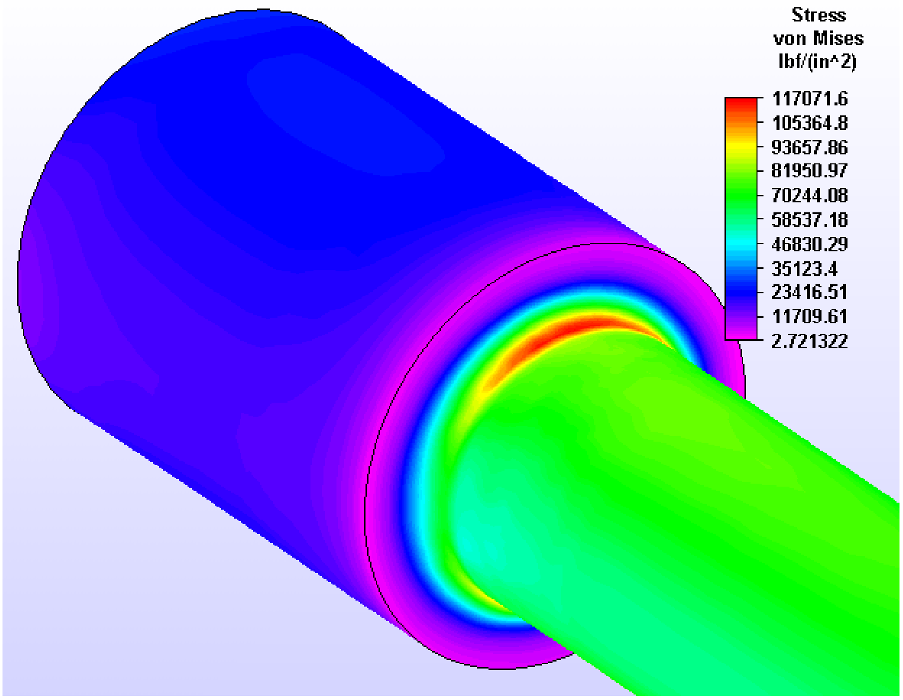
so we choose 1/n = 0.294, or n = 1/.294 = 3.40 = factor of safety against static failure.

16.0 Application of FEA Stresses

A major concern is how to apply the results of FEA derived stresses in the field of static failure analysis. You can't just blindly take the FEA results and use them, especially for ductile materials. FEA (if done properly) will determine the actual localized stresses in all of various discontinuities in a part. In the case of high stress gradients which are highly localized, like at a cross-hole in a bar or plate, **it would be overly conservative to fully apply those stress values when designing for static loading with ductile materials**. For brittle materials, on the other hand - you want to apply the full FEA results (corrected for notch sensitivity), and make sure that the mesh used is good enough to capture the elevated stresses in the critical areas.

FEA can effectively be used to determine the stress concentration factor, then the static failure analysis can be conducted by hand methods.

Take the torsion bar we looked at for the ductile static loading example, for instance. A good FEA on this part run at the maximum (failure) load we determined, 416.4 lbf, gave the following result in the area of the fillet:

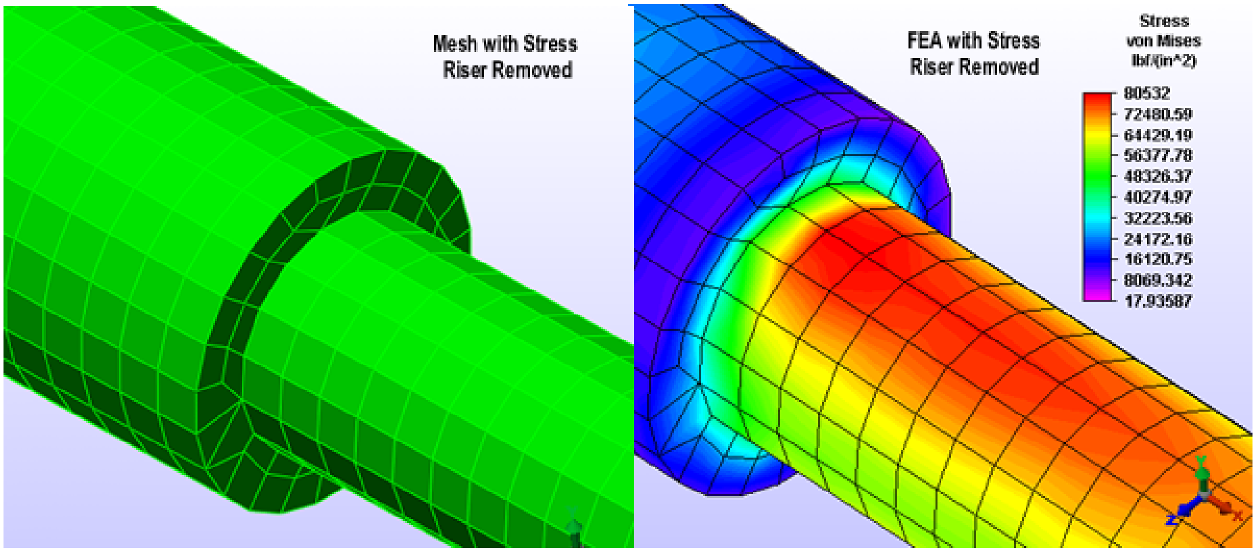


Obviously, stress concentration effects are showing in the FEA, which is reporting peak stress of 117 ksi, vs. the 81 ksi yield strength. What does a designer do about this? It depends on whether the material is ductile or brittle.

For ductile materials, we want to determine the nominal stress, since we won't be applying the stress levels that result from the stress concentration in the failure theory analysis.

One way to use the FEA results is to conduct a hand calculation if it is possible for the geometry, in conjunction with the FEA, and determine, at least approximately what the nominal stresses are in the net section of the part just before the high stress gradient effects of the stress concentration begin to appear. Or, if hand calculations are impractical, use the appearance of a high stress gradient to indicate the beginning of the stress concentration. A good indicator of high stress gradient in a contour plot is a very narrow band of color (like the yellow band in the previous figure). It may be helpful, if you know the approximate nominal stress to re-scale the contour colors around the nominal stress value to increase the resolution of the color bands. Once you know where the high stress gradient begins you can pick a node just outside that range and use that value for the stress for application to the failure theory, such as von Mises-Hencky.

Another option for ductile materials, (or to get a baseline for calculating K_t for brittle materials) is to eliminate the stress riser from the geometric model prior to the FEA meshing. This actually works well, and reduces analysis time for the FEA run.
Be sure when leaving out features that the overall stiffness of the part is not significantly affected. The results now show 80.5 ksi - very close agreement to failure stress determined by the failure theory calculations.



And one more option, which is effectively the same as the last option, is to intentionally use a course mesh in the area of the stress riser. If the mesh resolution is low enough to render the stress riser ineffective, then the elevated stresses associated with the discontinuity will never appear in the solution.

For brittle materials, use FEA results from a fully converged, and detailed mesh with a fine enough resolution to determine maximum stresses to determine K_t. Then use hand calculations, or one of the previously mentioned techniques to find the nominal stress just outside the range where the high stress gradient begins, then determine K_t from the ratio of the maximum stress from the high resolution mesh, and the nominal stress from the low resolution mesh or hand calculation. Then correct K_t for notch sensitivity and proceed using an applicable failure theory (usually modified Mohr's theory covers all brittle material cases, unless codes require something else).

In applying any of the previous methods, experience level and an understanding of the mechanics of materials is essential, as some judgment is involved. When eliminating details or using a course mesh, be sure that the mesh is still fine enough to fully capture nominal stress levels, and that the overall structure is not softened or stiffened significantly due to the changes in the mesh from the actual part geometry. This is particularly important if deflection is a major design consideration. In this case, the fully resolved mesh model can be used for deflection output, and another model with the stress riser removed by one of these methods can be used for the stress results.

Again, the good engineer never went wrong by verifying the FEA with hand calculations, and with plenty of tests on actual parts under actual or closely simulated conditions of use.

17.0 Bibliography

Collins, Jack A. "Failure of Materials in Mechanical Design." Second edition. New York: John Wiley & Sons, 1993.

Cook, R. "Finite Element Modeling for Stress Analysis." New York: John Wiley & Sons, Inc., 1995.

Lipson, C. "Basic Course in Failure Analysis" Cleveland: Penton, 1980.

Roark, R. & Young, W. "Formulas for Stress and Strain." New York: McGraw-Hill, 1982.

Shigley, J. & Mitchell, L. "Mechanical Engineering Design." New York: McGraw-Hill, 1983

Teng, Y. "How Linear FEA Helps in Fatigue Analysis." Machine Design; Penton Media - 2004.