PDHonline Course G428 (4 PDH)

Practical Application of Measurement Errors for Engineers & Surveyors

Instructor: Albert A. Avanesyan , PE, PLS.

2020

PDH Online | PDH Center

5272 Meadow Estates Drive
Fairfax, VA 22030-6658
Phone: 703-988-0088
www.PDHonline.com

An Approved Continuing Education Provider

Practical Application of Measurement Errors for Engineers & Surveyors

Albert A. Avanesyan , PE, PLS

1. INTRODUCTION

Human perception of the surrounding is going through the sensations experienced by our senses. When trying to estimate a distance we can easily misestimate a distance by 10% of its value (if distance is small), and by larger error (if distance is long). Same is true for the weight, temperature, humidity, etc. We normally cannot feel atmospheric pressure and we do not have the ability to feel electro-magnetic fields and radio signals.

In order to have the ability to accurately measure, different instruments, tools, and complicated devices are utilized.

However, all of these instruments, tools and devices designed and manufactured employ modern high precision technologies which are not perfect. Any measurements taken with such devices will have errors. True value of the measured value (a particular quantity, property, or condition) is hidden in most cases.

Therefore the question of lowering and eliminating influence of errors on the results of the measurements is highly significant.

In order to identify Blunt errors and eliminate them, most measurements are performed with repetitions.

Systematic errors can be identified and counted for in the final results. They are typically generated by the following factors:

Instrumental
Personal or Subjective
External Conditions of the Measurements (i.e. temperature, pressure, humidity, etc.)

However, the final results of the measurements will always include errors that are outside of our control and they are known as **Random Errors**.

Random Errors are unavoidable and will be part of any measurement. Random Errors cannot be eliminated, therefore in relation to the Random Errors there are two major questions:

How do we determine the accuracy of the measured results and what degree of trust do they deserve?
How to make most accurate conclusions for the measurement results in relation to their accuracy?

Theory of the Measurement Errors is providing answers to these questions.

2. PROPERTIES OF THE RANDOM ERRORS OF THE MEASUREMENTS

The Theory of Measurements Errors is a mathematical subject based on the four apparent properties of the random errors.

Definition of the random error Δ is a difference between a true value X and the measured value of a given measurement:

$$X - a = \Delta \qquad (2.1)$$

if $X > a$, Δ has a positive value;

if $X < a$, Δ has a negative value;

By adopting above conditions we can formulate main properties of the Random Errors:

1. Equal random errors (positive or negative) equally probable to occur and represented in the measurements with the same frequency;
2. In certain given conditions of the measurement (instrument, observer, weather, etc.) absolute value of the random errors may not exceed certain limit;
3. Random error with smaller absolute value will occur more frequently;
4. Mean value of the random errors of the measurements of the same measured property with same accuracy is approaching 0 with increasing number of the measurements n, i.e. when $n \Rrightarrow \infty$

First 3 properties of the random errors are obvious.
Below we will prove the fourth property of the Random Error.
Let's assume that we have a set n of the equal precision measurements

$$a_1, a_2, \ldots, a_n \qquad (2.2)$$

of the measured subject whose true value is X;

The Random Errors of these measurements are

$$\Delta_1, \Delta_2, \ldots, \Delta_n \qquad (2.3)$$

Having a large enough amount of n, (measurements) in (2.3) we can expect that the positive and negative errors will occur almost equally, and this will be more true when amount of the errors Δ is large.

When added, positive errors will be compensated by the negative once, thus sum of the all errors will remain as an end value and relatively small regardless the number of the n, measurements.

Therefore the following will be true:

$$\lim_{n\to\infty}\left(\frac{\Delta_1, \Delta_2, \ldots, \Delta_n}{n}\right) = 0$$

If [Δ] - sum of delta, then above equation is expressed as follows;

$\lim\limits_{n\to\infty}\left(\frac{[\Delta]}{n}\right)$ =0 proving 4th property of the random errors (2.4)

3. ARITHMETIC MEAN

Since

$X - a = \Delta$ see (2.1),

then

$X - a_1 = \Delta_1$
$X - a_2 = \Delta_2$
. . . .
$X - a_n = \Delta_n$ (3.1)

By adding left and right sides of the above equations we will have:

$nX - a_1 - a_2 - \ldots - a_n = \Delta_1 + \Delta_2 + \ldots + \Delta_n$, or

$nX - [a] = [\Delta]$,

$X = \frac{[a]}{n} + \frac{[\Delta]}{n} \Rightarrow \frac{[a]}{n}$ substitute with $\mathbf{x_0}$, and $\frac{[\Delta]}{n}$ with ξ, (3.2)

then:

$X = \mathbf{x_0} + \xi$ (3.3)

$\mathbf{x_0}$ is an arithmetic mean of the measured value X, and ξ is a random error of said arithmetic mean.

According to (2.4):

$\lim\limits_{n\to\infty}\xi = 0$, then $\lim\limits_{n\to\infty}\mathbf{x_0} = X$ (3.4)

Conclusion:

Arithmetic mean of the measurements performed with same instrument and observer in, equal atmospheric conditions or with equal weight of the measurements, is approaching the true value of the measured item when number of the measurements is infinite

Based on this conclusion, it is common to assume that arithmetic mean of the equal weight measurements of the same object is most reliable result of said measurements.

4. STANDARTD DEVIATION

Standard Deviation m can be defined a statistical measure of precision.
The following expression is describing a standard deviation:

$$+/-\sqrt{\frac{[\Delta_1\Delta_1]+[\Delta_2\Delta_2]+\cdots+[\Delta_n\Delta_n]}{n}} \quad \text{or} \quad m = +/-\sqrt{\frac{[\Delta\Delta]}{n}} \qquad (4.1)$$

A low standard deviation indicates that the data points tend to be very close to the mean, whereas high standard deviation indicates that the data are spread out over a large range of values.

standard deviation is rather best representing the accuracy of the performed measurements, worthiness of the obtained results and characterizing the conditions of the measurements.
This occurs due to the following:

Relatively big influence on the standard deviation is imposed by the large random errors, and this is degrading reliability of the measurements;
Standard deviation is a stable parameter and relatively small number of the measurements is enough to determine a standard deviation value with good accuracy;
We can judge about probable maximum error in the given particular conditions of the measurements.

Based on the theoretical research and practical experience the absolute value of the random error will not exceed triple standard deviation;

$$\Delta_{max} = 3\sigma \qquad (4.2)$$

5. STANDARD DEVIATION OF ARITHMETIC MEAN

Now we will determine standard deviation of arithmetic mean.
The value of X is measured in s-series with n-measurements in each series:

$a_{11}, a_{12}, \ldots, a_{1n}$

$a_{21}, a_{22}, \ldots, a_{2n}$

. . . .

$a_{s1}, a_{s2}, \ldots, a_{sn}$ (5.1)

The errors of these measurements are as follows:

$\Delta_{11}, \Delta_{12}, \ldots, \Delta_{1n}$

$\Delta_{21}, \Delta_{22}, \ldots, \Delta_{2n}$

. . . .

$\Delta_{s1}, \Delta_{s2}, \ldots, \Delta_{sn}$ (5.2)

From every measurement set we can determine arithmetic mean and express this as follows:

$x_{10}, x_{20}, \ldots x_{s0}$ (5.3)

and random errors of these arithmetic means according to (3.2) $\frac{[\Delta]}{n} = \xi$:

$\xi_1, \xi_2, \ldots \xi_s,$ (5.4)

Standard Deviation of the series of arithmetic means(5.3) is determined as follows:

$M = +/- \sqrt{\frac{[\xi\xi]}{s}}$ or $M^2 = \frac{[\xi\,\xi]}{s}$ (5.5)

Substituting $\frac{[\Delta]}{n} = \xi$ or $\xi n = [\Delta]$ from (3.2) and raising both sides of this equation to power 2, we will have the following:

$(\xi n)^2 = ([\Delta])^2$

$\xi^2 n^2 = [\Delta\Delta] + 2[\Delta_k \Delta_e]$ (5.6)

where $[\Delta_k \Delta_e]$ is a sum of the products of the pairs of random errors;

Applying equation (5.6) to (5.2), we will have:

$n^2(\xi_1)^2 = [\Delta 1\Delta 1] + 2[\Delta 1k\Delta 1e]$

$n^2(\xi_2)^2 = [\Delta 2\Delta 2] + 2[\Delta 2k\Delta 2e]$

. . . .

$n^2(\xi_s)^2 = [\Delta s\Delta s] + 2[\Delta sk\Delta se]$

by adding all series, we will have:

$n^2[\xi\,\xi] = [[\Delta\Delta]] + 2[[\Delta k\Delta e]]$ (5.7)

$[[\Delta\Delta]]$ of (5.7) is representing a sum of the ns added squares of all errors of the s-series (5.2), and when s is large (multiple series) according to equation (4.1)

$m = +/- \sqrt{\frac{[[\Delta\Delta]]}{ns}}$, where m is a standard deviation of the separate measurement obtained from the combination of series (5.1), so

$m^2ns = [[\Delta\Delta]]$ (5.8)

$2[[\Delta_k\Delta_e]]$ of (5.7) is consists from the sum of positive and negative numbers therefore when s (number of series) is large, the $2[[\Delta_k\Delta_e]]$ member of the equation (5.7) becomes very insignificant compared to $[[\Delta\Delta]]$ of the same equation and is negligible, therefore:

$n^2[\xi\,\xi] = [[\Delta\Delta]]$, or according to (5.8)

$n^2[\xi\,\xi] = m^2ns.$

After dividing both sides to n^2s, we will have:

$\frac{[\xi\,\xi]}{s} = \frac{m2}{n}$ and according to (5.5), Type equation here.

$M^2 = \frac{m2}{n}$, or

$$M = \frac{m}{\sqrt{n}} \qquad (5.9)$$

Conclusion:

Standard deviation of arithmetic mean of the measurements M performed in same conditions (with equal weight) is inversely proportional to the square root of the number of measurements n;

6. MOST PROBABLE ERRORS AND THEIR APPLICATION TO CALCULATE STANDART DEVIATION

Formula $m = +/- \sqrt{\frac{[\Delta\Delta]}{n}}$ (4.1) of the standard deviation

is rarely used in practical application since it has embedded random errors of the measurements and the true size of these errors is not known

Below we will derive the formula of the standard deviation for practical application:

Let's assume that we have one set of the measurements:

$a_1, a_2, \ldots, a_n$

with random errors

$\Delta_1, \Delta_2, \ldots, \Delta_n$

According to (2.1)

$X - a_1 = \Delta_1$

$X - a_2 = \Delta_2$

. . . .

$X - a_n = \Delta_n$ (6.1)

Difference between arithmetic mean x_0 and each given measurement accordingly we will designate with

δ_1 , δ_2 , ... δ_n ,

$$x_0 - a_1 = \delta_1$$
$$x_0 - a_2 = \delta_2$$
$$\cdots\cdots$$
$$x_0 - a_n = \delta_n \qquad (6.2)$$

Since arithmetic mean represents most probable result of the measurements then values of δ_1 , δ_2 , ... δ_n , are commonly called as the most probable errors of the measurements. Random errors $\Delta_1, \Delta_2, \ldots, \Delta_n$ are called true errors of the measurement .

Subtracting series (6.2) from the series (6.1) we will have following:

$$X - x_1 = \Delta_1 - \delta_1$$

$$X - x_0 = \Delta_2 - \delta_2$$

$$\cdots\cdots\cdots$$

$$X - x_0 = \Delta_n - \delta_n \qquad (6.3)$$

Since $X = x_0 + \xi$ or $X - x_0 = \xi$, as shown in (3.3), the equation (6.3) can be rewritten as follows:

$$\Delta_1 = \delta_1 + \xi$$
$$\Delta_2 = \delta_2 + \xi$$
$$\cdots\cdots\cdots$$
$$\Delta_n = \delta_2 + \xi \qquad (6.4)$$

or

$$\Delta_1^2 = \delta_1^2 + \xi^2 + 2\,\xi\,\delta_1$$
$$\Delta_2^2 = \delta_2^2 + \xi^2 + 2\,\xi\,\delta_2$$
$$\cdots\cdots\cdots$$
$$\Delta_n^2 = \delta_n^2 + \xi^2 + 2\,\xi\,\delta_n$$

and after adding all series we will have following:

$$[\Delta^2] = [\delta^2] + n\xi^2 + 2\,\xi\,[\delta] \qquad (6.5)$$

also, by adding left and right sides of the equation (6.2), the result will be as follows:

$$nx_0 - [a] = [\delta]$$

and since $x_0 = \frac{[a]}{n}$ as shown in (3.2),

then $[\delta] = 0$ (6.6)

Thus sum of the most probable errors of the measurements performed with equal accuracy (same weight of the measurements) in any n number of the measurements is equal to 0

Based on this statement an equation (6.5) will have a following form:
$[\Delta^2] = [\delta^2] + n\xi^2$ or after dividing both sides of this equation to n:

$\frac{[\Delta 2]}{n} = \frac{[\delta 2]}{n} + \xi^2$ or according to (4.1), where $m = +/- \sqrt{\frac{[\Delta 2]}{n}}$,

$m^2 = \frac{[\delta 2]}{n} + \xi^2$ (6.7)

When n is relatively large, the value if the $\frac{[\delta 2]}{n}$ is fairly stable, whereas ξ is decreasing with

increased n therefore in equation (6.7), ξ^2 as a secondary member can be substituted with its mean value equal to M^2:

per (5.6) $\xi^2 n^2 = [\Delta\Delta] + 2[\Delta_k\Delta_e]$, where $2[\Delta_k\Delta_e] \Rightarrow 0$ due to Δ's having positive and negative signs randomly , then $\xi^2 = \frac{[\Delta\Delta]}{n^2} = M^2$

then equation (6.7) can be expressed as follows:

$m^2 = \frac{[\delta 2]}{n} + M^2$,

but since $M = \frac{m}{\sqrt{n}}$ as shown in (5.9) then:

$m^2 = \frac{[\delta 2]}{n} + \frac{m2}{n}$, or after simplifying, the final equation will be as follows:

$$\boxed{m = +/- \sqrt{\frac{[\delta 2]}{n-1}}} \qquad (6.8)$$

Example:

Same angle has been measured 9 times in equal conditions (same instrument and observer, equal atmospheric conditions).

Table 1:

Measurement No.	Measured Angle a	δ	δ^2
1	2	3	4
1	57° 09' 31".1	+ 0".7	0.49
2	57° 09' 35".3	- 3".5	12.25
3	57° 09' 29".2	+2".6	6.76
4	57° 09' 31".7	+ 0".1	0.01
5	57° 09' 34".7	- 2".9	8.41
6	57° 09' 30".9	+ 0".9	0.81
7	57° 09' 31".7	+ 0".1	0.01
8	57° 09' 29".5	+ 2".3	5.29
9	57° 09' 32".1	- 0".3	0.09
Arithmetic Mean X_0 = 57° 09' 31".800		Σ δ =0".000	Σ δ^2=34.120

Arithmetic Mean of the 9 measurements is equal :

$$X_0 = \frac{[a]}{n} = 57° \; 09' \; 31.800$$

$$\delta_n = X_0 - a$$

Using $m = +/- \sqrt{\frac{[\delta 2]}{n-1}}$ (6.8)

and per (5.9) $M = \frac{m}{\sqrt{n}}$

we will have the following results:

$$m = +/- \sqrt{\frac{34.120}{8}} = +/- \sqrt{4.265} = +/- 2''.065, \text{ so}$$

$m = +/-2''.065$ $M = +/- \frac{2''.065}{\sqrt{9}} = +/- \frac{2''.065}{3} = +/- 0.688''$

According to (4.2) $\Delta_{max} = 3\ \delta = 3 \times 2''.065 = 6''.195$ and when compared

with actual errors from the column 3 of the Table 1, we can see that all errors are smaller than

$$\Delta_{max} = 6''.195.$$

7. PROBABLE ERROR

Probable Error r definition takes place in the theory of the measurement errors (do not confuse with Most Probable Error of the measurements).

Probable Error r is defined as follows:

Probable Error is a value of the random error in relation to which in the certain given conditions all random errors larger than Probable Error occur as often as all random errors smaller than Probable Error, i.e. with exactly same frequency.

From the math-statistics analysis, it was established that the Probable Error is approximately equal to $\mathbf{2/3}$ of the standard deviation value:

$$r = \frac{2}{3} m, \qquad \text{more precisely,} \quad \mathbf{r = 0.6745m} \qquad (7.1)$$

Example:

Table 2:

No.	Δ in seconds	[ΔΔ]	
1	9.2	84.64	
2	1.4	1.96	
3	-6.9	47.61	
4	9.7	94.09	
5	-3.9	15.21	
6	9.5	90.25	
7	-10.4	108.16	
8	-9.8	96.04	
9	-10.2	104.04	
10	1.5	2.25	
11	3.7	13.69	
12	2.2	4.84	
13	3.1	9.61	
14	-4.9	24.01	
15	0.9	0.81	
16	-0.2	0.04	
17	5.1	26.01	
	[Δ] = 0	[Δ Δ] = 723.26	

Since m = +/- $\sqrt{\frac{[\Delta 2]}{n}}$ =+/- $\sqrt{\frac{723.26}{17}}$ =+/- $\sqrt{\mathbf{42.545}}$ = +/- 6''.523, and based on (7.1) we can calculate r:

r = +/- 0.6745m =+/- 6".5 * 6.6745 =+/- 4".399

Let's write all Δ^s absolute values in the increasing order:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0.2	0.9	1.4	1.5	2.2	3.1	3.7	3.9	4.9	5.1	6.9	9.2	9.5	9.7	9.8	10.2	10.4

The random error Δ = 4".399 is lying in the middle of the series of measured random errors.

There are 8 errors that are smaller than 4".399 and 8 errors that are larger than 4".399, therefore the **Δ = 4".4 is defining an approximate value of the probable error for measurements** represented in Table 2. In order to obtain more accurate value of the probable error, we need to have very large amount of measurements.

8. ESTIMATING AN ACCURACY OF THE RESULT OF A CERTAIN FUNCTION WITH GIVEN ACCURACIES OF THE ALL ARGUMENTS OF THIS FUNCTION

One of the important tasks of the Theory of the Measurements Errors is defining the rules for estimating accuracy of the measured results .

We now know how to estimate an accuracy of the arithmetic mean series of measurements performed under the same conditions.

Now we will establish formulas for the estimating accuracy of different functions of the measured quantities.

Such estimate always takes place when we need to estimate an accuracy of the result of a certain function with given accuracies of the all arguments of this function.

In order to do that let's analyze following simple functions:

1. ***We are given the following function:***

$Y = kX$, (8.1)

where K is constant number.

Let's assume that in order to obtain value X, the following measurements are taken under the same conditions:

$a_1, a_2, \ldots, a_n$

with random errors

$\Delta_1, \Delta_2, \ldots, \Delta_n$ (8.2)

then, in order to find Y, we simply obtain series of calculated values as follows:

$$ka_1,\ ka_2, \ldots,\ ka_n \qquad (8.3)$$

The random errors of the series (8.3) members we can designate accordingly as follows:

$$\alpha_1,\ \alpha_2, \ldots, \alpha_3, \qquad (8.4)$$

By the definition, standard deviation of function Y will be as follows:

$$m_y = +/- \sqrt{\frac{[\alpha 2]}{n}} \qquad (8.5)$$

In order to find values of the random errors in series (8.4), using equation (2.1) we will have following:

$$\alpha_1 = Y - ka_1 = kX - ka_1 = k(X-a_1) = k\,\Delta_1$$
$$\alpha_2 = Y - ka_2 = kX - ka_2 = k(X-a_2) = k\,\Delta_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\alpha_n = Y - ka_n = kX - ka_n = k(X-a_n) = k\,\Delta_n$$

According (8.5) and above equations, we will have the following:

$$m_y = +/- \sqrt{\frac{[\alpha 2]}{n}} = +/- k\sqrt{\frac{[\Delta 2]}{n}}$$

and since standard deviation of X is defined per (4.1) as :

$$m_x = +/- \sqrt{\frac{[\Delta 2]}{n}}\text{, then}$$

$$m_y = km_x \qquad (8.6)$$

2. We are given the following function:

$$Y = X+Z \qquad (8.7)$$

Let's assume that in order to obtain value X, the following measurements are taken under the same conditions:

$a_1, a_2, \dots, a_n$ (8.8)

with random errors

$\Delta_{11}, \Delta_{12}, \dots, \Delta_{1n}$ (8.9)

and in order to obtain value Z, the following measurements are taken under the same conditions:
$b_1, b_2, \dots, b_n$ (8.10)

with random errors

$\Delta_{21}, \Delta_{22}, \dots, \Delta_{2n}$ (8.11)

standard deviations of series of (8.9) and (8.11) are as follows:

$$m_x = +/- \sqrt{\frac{[\Delta 1\ \Delta 1]}{n}} \qquad (8.12)$$

$$m_z = +/- \sqrt{\frac{[\Delta 2\ \Delta 2]}{n}} \qquad (8.13)$$

m_x & m_z are describing the conditions of measurements of X & Z.

Based on (8.7) , (8.8), (8.10) we can calculate m_y for the function Y as follows:

$(a_1 + b_1)$, $(a_2 + b_2)$, . . . , $(a_n + b_n)$ (8.14)

The random errors for series (8.14) we will designate as:

$\Delta_1, \Delta_2, \ldots, \Delta_n$ (8.15), then:

$$m_y = +/- \sqrt{\frac{[\Delta 2]}{n}} \qquad (8.16)$$

Random errors for series (8.14) are calculated as follows:

$\Delta_1 = Y - (a_1 + b_1) = X + Z - (a_1 + b_1) = (X - a_1) + (Z - b_1) = \Delta_{11} + \Delta_{12}$;

$\Delta_2 = Y - (a_2 + b_2) = X + Z - (a_2 + b_2) = (X - a_2) + (Z - b_2) = \Delta_{21} + \Delta_{22}$;

. . . .

$\Delta_n = Y - (a_n + b_n) = X + Z - (a_n + b_n) = (X - a_n) + (Z - b_n) = \Delta_{n1} + \Delta_{n2}$;

Square both sides of the above equations:

$\Delta_1^2 = \Delta_{11}^2 + \Delta_{12}^2 + 2\,\Delta_{11}\Delta_{12}$;

$\Delta_2^2 = \Delta_{21}^2 + \Delta_{22}^2 + 2\,\Delta_{21}\Delta_{22}$;

. . . .

$\Delta_n^2 = \Delta_{n1}^2 + \Delta_{n2}^2 + 2\,\Delta_{n1}\Delta_{n2}$;

by adding these equations, we will get following:

$[\Delta^2] = [\Delta_1^2] + [\Delta_2^2] + \mathbf{2[\Delta_1\Delta_2]}$ and after dividing to n

$$\frac{[\Delta^2]}{n} = \frac{[\Delta_1^2]}{n} + \frac{[\Delta_2^2]}{n} + \frac{2[\Delta_1\Delta_2]}{n} \qquad (8.17)$$

Since each $\Delta_{n1}\Delta_{n2}$ is the product of errors that are different and can be negative and positive, by the definition of the random errors these products $\Delta_{n1}\Delta_{n2}$ have all the properties of the Random Errors, and therefore when n is large, the value of the $\frac{2[\Delta 1\Delta 2]}{n}$ is close to zero and negligible.

Then:

$$m_y^2 = m_x^2 + m_z^2 \qquad (8.18)$$

For the equation

$$Y = X - Z \qquad (8.19)$$

$$\frac{[\Delta^2]}{n} = \frac{[\Delta_1{}^2]}{n} + \frac{[\Delta_2{}^2]}{n} - \frac{2[\Delta_1\Delta_2]}{n} \qquad (8.20)$$

$\frac{2[\Delta_1\Delta_2]}{n}$ = 0 as explained above, therefore

$$m_y{}^2 = m_x{}^2 + m_z{}^2 \qquad (8.21) \text{ which equals } (8.18)$$

Conclusion:

for $Y = X +/- Z$ $m_y{}^2 = m_x{}^2 + m_z{}^2$ is used to find standard deviation of the function Y

3. ***We are given the following function:***

$$Y = X + Z + T \qquad (8.22)$$

Let substitute X+Z with U;

$$U = X + Z \qquad (8.23) \quad \text{then}$$

$$Y = U + T \qquad (8.24)$$

According to (8.18) we can say

$$m_y{}^2 = m_u{}^2 + m_t{}^2 \qquad (8.25)$$

and for (8.23):

$m_u{}^2 = m_x{}^2 + m_z{}^2$, by substituting m_u into (8.25) we will have:

$$m_y{}^2 = m_x{}^2 + m_z{}^2 + m_t{}^2 \qquad (8.26)$$

4. ***In case of the general function:***

$$Y = X +/- Z +/- T +/- \ldots +/- W \qquad (8.27)$$

$$m_y{}^2 = m_x{}^2 + m_z{}^2 + m_t{}^2 + \ldots + m_w{}^2 \qquad (8.28)$$

If the standard deviation m is equal for all measurements, then:

$$m_y = m\sqrt{n} \qquad (8.29)$$

5. ***We have the following function:***

$$Y = k_1 X +/- k_2 Z +/- \ldots +/- k_n W \qquad (8.30),$$

Bu applying following designations:

$k_1 X = X_1$

$k_2 Z = Z_1$

. . . . . .

$K_n W = W_1$ (8.31)

We can express (8.30) as follows:

$Y = X_1 +/- Z_1 +/- \ldots +/- W_1$,

using (8.28) we will be able to calculate $m_{y;}$

$m_y = m_{x1}^2 + m_{z1}^2 + \ldots + m_{w1}^2$ (8.32)

Considering designations of (8.31) and based on the formulas

$Y = kX,$ (8.1)

$m_y = km_x$ (8.6)

we will have:

$m_{x1} = k_1 m_x$

$m_{z1} = k_2 m_z$

. . . .

$m_{w1} = k_n m_w$ then (8.32) can be expressed as follows:

$m_y = k_1^2 m_x^2 + k_2^2 m_z^2 + \ldots + k_n^2 m_w^2$ (8.33)

6. *Now we will analyze a General Function:*

$Y = f(X, Z, \ldots W)$ **(8.34)**

Where *X, Z, . . . , W are True Values of actually measured items (i.e. angles, lengths, temperatures, etc.)*

The results of these measurements are :

a, b, . . . , l *and their standard deviations are accordingly* $m_x, m_z, \ldots, m_w$

Since we do not know True Values X, Z, . . . , W in the formula (8.34) lets substitute them with the results of the measurements *a, b, . . . , l;*

By doing this we will obtain calculated result for function (8.34), which we will designate as q, so:

$q = f(a, b, \ldots, l)$

In order to estimate an accuracy of obtained result of q, we will apply Taylor polynomial *for the*

$$Y = f(X, Z, \ldots W) \qquad (8.34)$$

$x_0, z_0, \ldots w_0$ are arguments and $\alpha, \beta, \ldots, \lambda$ their remainders, then: .

$$Y = f\{(x_0 + \alpha), (z_0 + \beta), \ldots, (w_0 + \lambda)\} = f(x_0, z_0, \ldots w_0) + \frac{df}{dx0}\boldsymbol{\alpha} + \frac{df}{dz0}\boldsymbol{\beta} + \frac{df}{dw0}\boldsymbol{\lambda} + \ldots..$$

By designating $y_0 = f(x_0, z_0, \ldots w_0)$ and $\varepsilon = (Y - y_0)$ and assuming that remainders are small and negligible, we can limit calculations with the first degree of polynomial *and have the following result:*

$$\varepsilon = \frac{df}{dx0}\boldsymbol{\alpha} + \frac{df}{dx0}\boldsymbol{\beta} + \frac{df}{dw0}\boldsymbol{\lambda} \qquad (8.35)$$

Formula (8.35) can be applied to calculate adjustment to an approximate value of the function y_0 in order to obtain its True Value.

For that it is necessary to know true values of the adjustments for the approximate values of the arguments $\alpha, \beta, \ldots, \lambda$. and we will need to know true values of the arguments X, Z, . . . , W.

We are only given the measured values of the arguments and standard deviations of these measurements :

a, b, . . . , l *and their standard deviations are accordingly* $m_x, m_z, \ldots, m_w$

it is clear that adjustments $\alpha, \beta, \ldots, \lambda$ will have same standard deviations as measurements themselves.

By calculating adjustments ε for the approximate value of y_0 using such adjustments and adding it to y_0, we will find a Calculated Value of the Function q.

Accuracy of True Value of the function Y will depend on the accuracy of Calculated Value of the Function q, which is as accurate as the value of the ε, therefore a standard deviation of the ε is equal to the standard deviation of the Y:

$$m_\varepsilon = m_y$$

According to (8.35) an adjustment ε is a Linear Function of adjustments $\alpha, \beta, \ldots, \lambda$, in which partial differentials

$\frac{df}{dx0}, \frac{df}{dx0}, \frac{df}{dw0}$ are playing role of the coefficients $k_1, k_2, \ldots, k_n$ and by applying them to the formula

$$(8.33) \qquad m_y = k_1^2 m_x^2 + k_2^2 m_z^2 + \ldots + k_n^2 m_w^2$$

we will get the following result:

$$m_y^2 = \left(\frac{df}{dx0}\right)^2 m_x^2 + \left(\frac{df}{dz0}\right)^2 m_z^2 + \ldots + \left(\frac{df}{dw0}\right)^2 m_w^2 \quad (8.36)$$

9. ROUNDING ERRORS DURING MEASUREMENTS

Any measurement is concluded by the observers final result recordation.

Such final recorded result always will have embedded Rounding Error.

The maximum error of the rounding will be equal to $\frac{1}{2}$ of the order at which a measurement has been performed. If measurement reading of the distance is performed with the accuracy to 0.01', then the maximum rounding error will not exceed 0.005', and if angle reading is performed with the accuracy of 5", then the maximum rounding error will not exceed 2".5 etc.

Further, all properties of the random errors are also applicable to rounding errors i.e. all absolute values of the rounding errors are smaller than the maximum rounding error and equally frequent.

The only relation between standard deviation and a maximum rounding error as well as with the probable error will be different in this case.

For establishing this relation let's assume that a rounding error by changing with defined limits ε has maximum value α;

So α is consisting of ε occurring n times, i.e:

$$\alpha = \varepsilon n \qquad \text{or} \qquad \varepsilon = \frac{\alpha}{n} \qquad (9.1)$$

we can express a series of different values that absolute value of rounding error can take place:

$$0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots, n\varepsilon \qquad (9.2)$$

This series have (n+1) numbers.

Since all errors of series (9.2) are equally probable, then with large number of the measurements n, each of them must occur with equal frequency, therefore for determination of the standard deviation of the rounding error will be sufficient to use series (9.2)

$$m = +/- \sqrt{\frac{\varepsilon^2 + (2\varepsilon)^2 + (3\varepsilon)^2 + \ldots + (n\varepsilon)^2}{n+1}}$$

$$m = +/- \varepsilon \sqrt{\frac{(1)^2 + (2)^2 + (3)^2 + \ldots + (n)^2}{n+1}}$$

$$1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$m = +/- \varepsilon \sqrt{\frac{n(2n+1)}{6}}$$

and since $\varepsilon = \frac{\alpha}{n}$ (9.1), then:

$$m = +/- \alpha\sqrt{\frac{2n+1}{6n}}$$

$$m = +/- \alpha\sqrt{\left(\frac{1}{3} + \frac{1}{6n}\right)}$$

with n $\rightarrow \infty$ $m = +/- \frac{\alpha}{\sqrt{3}}$ or

$\alpha = +/- m\sqrt{3} = +/- 1.7m$ $\alpha = +/- 1.7m$ (9.3)

Conclusion: If rounding error is large in comparison to errors due to the other factors, the maximum combined error of the result of the measurement in general will be less than triple standard deviation, i.e. :

$$\Delta_{max} < 3m \qquad (9.4)$$

On the other hand, since small errors are much more frequent than the large ones, then per (9.3)

$$\Delta_{max} > 2m \qquad (9.5)$$