PDHonline Course L121 (4 PDH)

Vertical Curves

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2020

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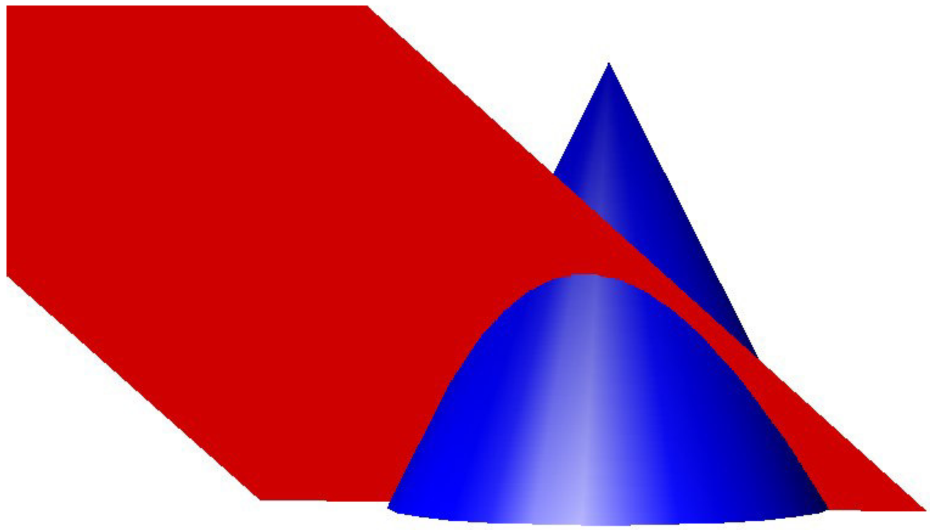

Figure 1

Introduction

A parabola is a conic section. It's the curve that can be obtained by a plane intersecting a right circular cone parallel to the side (elements) of the cone. Surveyors and engineers have traditionally used the term *vertical curve* to describe the use of parabolic curves in design work. They are used for transitions from one straight grade to another in designing crowns for pavements, routes such as highways and railways, as well as dam spillways, landscape design and of course roller coasters.

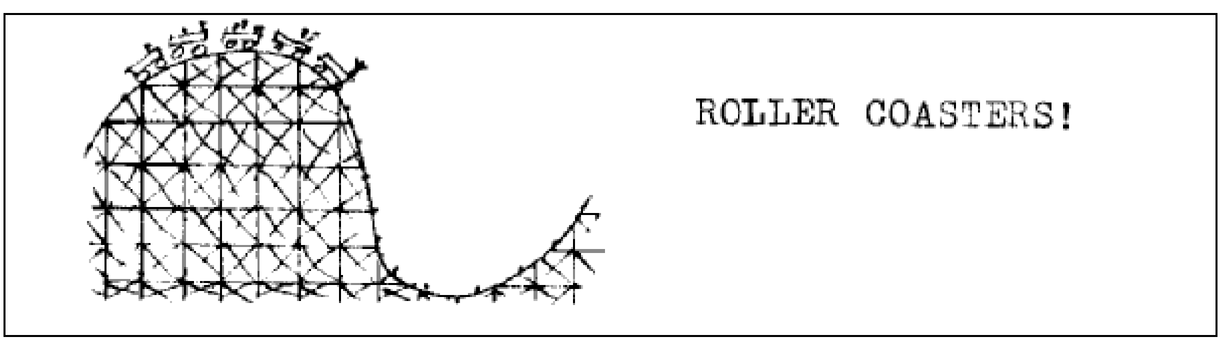

Figure 2

Most often vertical curves are used to improve the comfort, safety and appearance of routes. They are just as crucial to good work as horizontal curves.

However, unlike a circular curve a vertical curve does not have a constant radius at all points along the curve. Nevertheless it does have a quality that is very useful indeed. This quality is expressed by the general equation of a parabola:

$$y = ax^2$$

In a vertical curve the offsets from a tangent to the curve always vary as the square of the distance along the tangent from the point of tangency.

Common Properties of Vertical Curves

A smooth transition between constant grades is necessary because if it were not ameliorated the *rate of change, r,* would be too abrupt. In fact, it is only when the algebraic difference of the intersecting grades is less than one-half of 1 percent that vertical curves become unnecessary. If the grade change were allowed to all take place in one spot it would make for a bumpy ride.

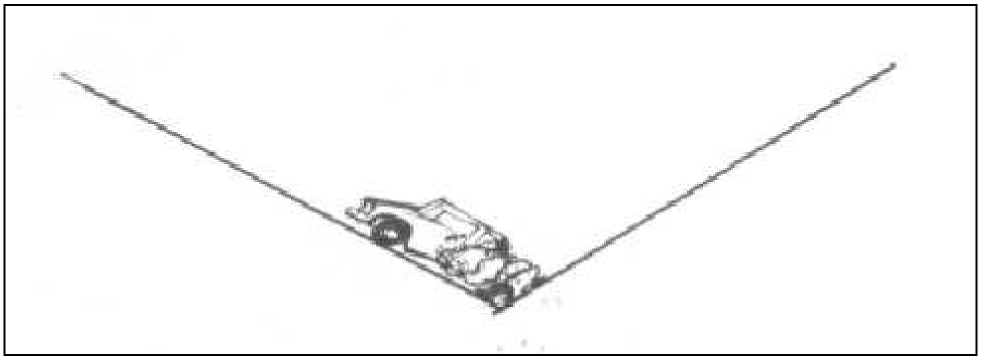

Figure 3

Wouldn't want to turn that brand new Toyota into a corn flake.

Now, any number of curves might work for this smoothing out the transition. For example, a circular curve would be feasible, but it has the weakness that for equal increments along the horizontal axis (a,b,c, & d) the corresponding vertical distances are unequal (w,x,y,&z) and inconvenient to calculate.

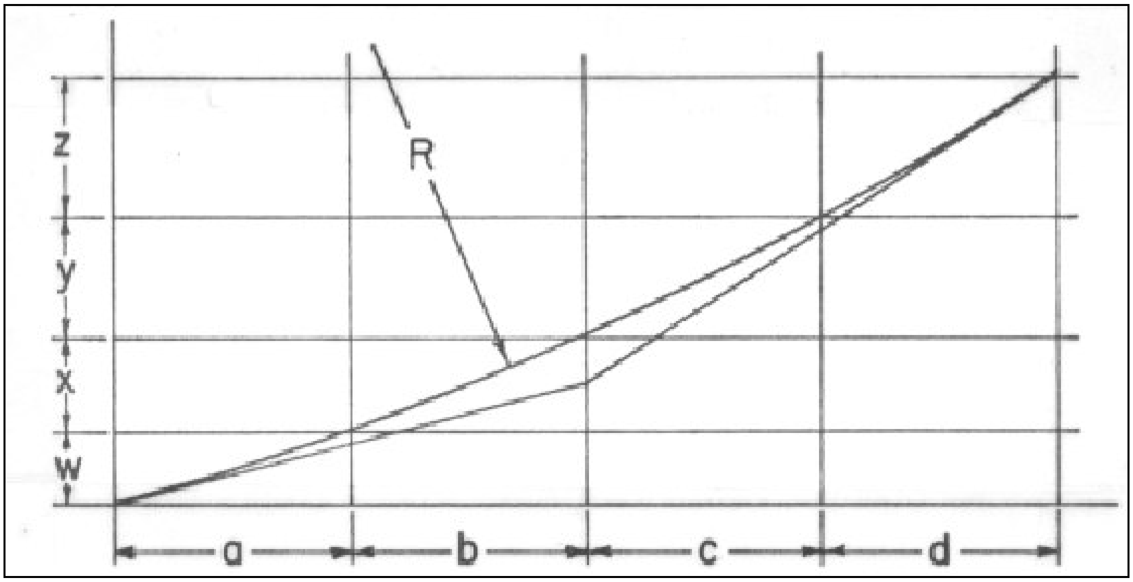

Figure 4

The most significant strength of the vertical curve is its constant rate of change. In fact, this constant rate of change is the key to the calculation of a vertical curve. Offsets from the tangent to a the vertical curve, the *tangent offsets*, are proportional to the squares of the distances from the point of tangency. That point of tangency may be the *point of vertical curvature, PVC,* or *the point of vertical tangency, PVT.*

Further all tangent offsets are considered to be measured vertically from the straight grade lines, the tangents to the vertical curve. These tangent offsets will be considered positive if directed upward, negative if directed downward from the intersecting grade lines. And the offsets from the two straight grade lines are symmetrical with respect to the PVI.

Please note that throughout this discussion all distances along the vertical curves will be measured horizontally. And the length of a vertical curve is always its horizontal projection. There is, of course, error inherent in these conventions. Nevetheless, since vertical curves are actually quite flat, in practice, that error is negligible.

The vertex, PVI of an equal-tangent, that is a symmetrical, vertical curve is midway between the beginning of the vertical curve, that is the point of vertical curvature PVC, and its end, the point of vertical tangency PVT. The center of the vertical curve is midway between the vertex, the PVI, and the long chord. In other words, the external distance in a symmetrical vertical curve always equals the middle ordinate.

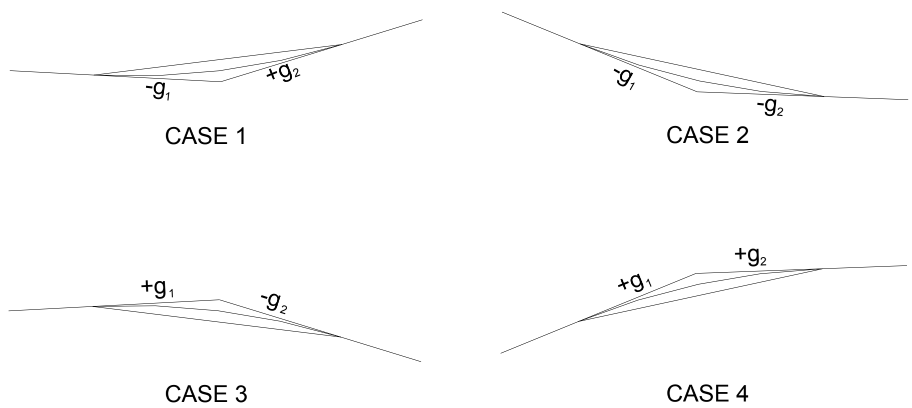

Figure 5

Here are just four of many possible arrangements of vertical curves. Cases 1 and 2 are *sag* vertical curves and Cases 3 and 4 are *crest*, or *summit* vertical curves. If the *tangent offsets* are subtracted if the curve is below the PVI, and hence offsets lessen the elevations of the stations along the grade tangents. If the *tangent offsets* are positive, the curve is above PVI, and hence offsets increase the elevations of the stations along the grade tangents.

Factors Governing the Length of Vertical Curves.

When calculating a vertical curve if the given parameters only include the entering and exiting grades the rate of change, r, or the length of the vertical curve, L, must be assumed. The PVI is usually placed at a full station, but that is not necessarily the case in road work.

The L or the length of a vertical curves over crests or into sags are generally governed by the algebraic difference of the intersecting grades. In the design of highways the controlling factor is usually visibility. In railroads it is rate of change of grade that is more critical.

Vertical curves should have sufficient length to hold the rate of change of grade to a minimum. When this goal is achieved efficiency, smooth riding, and pleasing appearance are maximized. For example in the Burlington Northern Santa Fe Design Guidelines for Industrial Track Projects, March 2004, page 4 it is stated, "Vertical curves must be provided at break points in profile grade. The rate of change shall not exceed 2.0 in summits or sags." On highways, the length of a vertical curve is governed by sight distances based on a consideration of the speed and safety of traffic.

Vertical Curve Notation

The notation used in this course will include:

1) L, as the length of a vertical curve. It can be expressed in 100-ft stations or in feet but is always measured along the horizontal.
2) The two grades are expressed in the direction of stationing. The entering grade is g_1, and the exiting grade is g_2
3) The total change in grade through the vertical curve is $(g_2 - g_1)$.
4) The rate of change is r. It can either be calculated in terms of feet or 100-ft stations. In both cases it is found by dividing the total change in grade by the length of the vertical curve. $\frac{(g_2 - g_1)}{L} = r$
5) The vertex, where the two straight grade lines meet is the point of vertical intersection, *PVI.*
6) The beginning of the vertical curve is the point of vertical curvature, *PVC.*
7) The end of the vertical curve is the point of vertical tangency, *PVT.*
8) The elevation of the beginning of the vertical curve , the PVC is e
9) The distance along the vertical curve to the station for which the elevation is desired is x
10) The elevation of the station desired is y

Calculation Methods

There are two methods for determining elevations along vertical curves that will be presented here. Either method is applicable to both highway and railroad profiles. They are:

1) Method A, where offsets are calculated in terms of a proportion per distance and elevation.
2) Method B, where offsets are calculated in terms of a quadratic formula per r, that is the *rate of change* of grade. This method is a bit more convenient in those instances when it is necessary to calculate the elevation of irregular stations on a vertical curve

Method A

As mentioned earlier the offsets from a tangent to a vertical curve, or any parabola for that matter, vary as the square of the distance from the point of tangency. That point of tangency can either be at the beginning of the vertical curve, the PVC, or at the end, the PVT. Therefore, all intermediate tangent offsets from the straight grade lines to the vertical curve can be computed in terms of what might be called the central offset, that is the distance from the PVI to the middle of a symmetrical vertical curve. In other words, there is a proportion there that can be very useful as you will see.

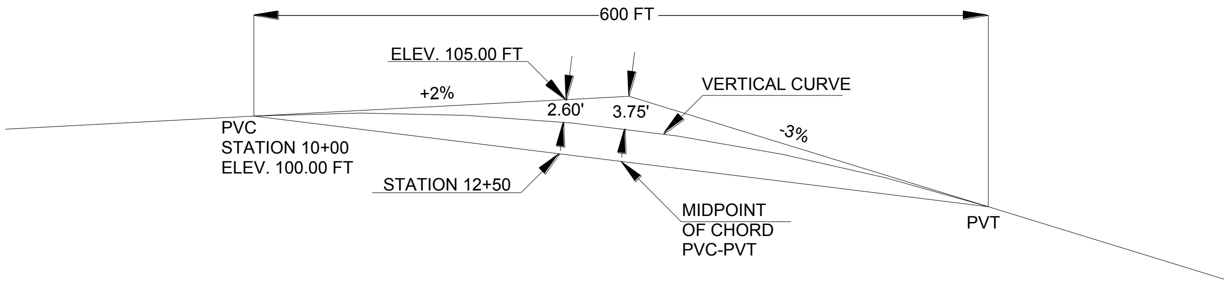

Figure 6

The objective is to find the elevation of Station 12+50 in the vertical curve illustrated here.

First, find the elevations of the PVI and the PVT

The length on either side of the PVI for a symmetrical vertical curve is:

$$\frac{L}{2}$$

where:
L = the full length of the vertical curve

Therefore:

$$\frac{600\,ft}{2} = 300\,ft$$

The station of the PVI station is then:

$$(10+00) + (3+00) = 13+00$$

And since this is a symmetrical vertical curve the elevation of the PVI is:

$$\text{PVI elevation} = \text{PVC elevation} + (g_1)\ (\text{distance PVC-PVI})$$

where:
g_1 = is the entering grade

Therefore:

$$\text{PVI elevation} = (100.00\text{ ft}) + (+0.02)(300.00\text{ ft}) = 106.00\text{ ft}$$

The elevation of the PVT is:

$$\text{PVT elevation} = \text{PVI elevation} + (g_2)\text{ (distance PVI-PVT)}$$

where:
g_2 = is the exiting grade

Therefore:

$$\text{PVT elevation} = (106.00\text{ ft}) + (-0.03)(300.00\text{ ft}) = 97.00\text{ ft}$$

Second, find the elevation of the midpoint of the the chord from the PVC to the PVT:

In a symmetrical vertical curve this elevation is simply the average of the elevation of the PVC and the PVT

$$\text{Elev. of the midpoint of the chord PVC-PVT} = \frac{ElevationPVC + ElevationPVT}{2}$$

Therefore:

$$\text{Elev. of the midpoint of the chord PVC-PVT} = \frac{100\,ft + 97\,ft}{2} = 98.50\,ft.$$

Third, find the elevation of the midpoint of the vertical curve:

In a symmetrical vertical curve the elevation of the midpoint of the vertical curve is midway between the elevation of the midpoint of the PVC – PVT chord and the elevation of the PVI.

$$\text{Elev. of the midpoint of the vertical curve} = \frac{ElevationChordMP + ElevationPVI}{2}$$

Therefore:

$$\text{Elev. of the midpoint of the vertical curve} = \frac{106\,ft + 98.5\,ft}{2} = 102.25\,ft.$$

Fourth, find the tangent offset from the PVI to the vertical curve. It is the difference between the elevation of the PVI and the elevation of the midpoint of the vertical curve. In this case:

Elevation of the PVI	=	106.00 ft
Elevation of the midpoint of the vertical curve	=	-102.25 ft
		3.75 ft

Fifth, find the tangent offset for the desired station, 12+50. From the tangent line, that is the straight grade line, to the vertical curve at any station, the tangent offsets vary with the square of the distance from the PVC or PVT. The following relationship applies for the vertical curve in the Figure 5 above:

$$\frac{\text{tangent offset at the desired station}}{(\text{distance from PVC})^2} = \frac{\text{tangent offset at PVI}}{(\text{distance from PVC})^2}$$

Therefore:

$$\frac{\text{tangent offset at 12+50}}{(250)^2} = \frac{\text{tangent offset at 13+00}}{(300)^2}$$

The tangent offset at 13+00 is 3.75 feet, therefore:

$$\frac{\text{tangent offset at 12+50}}{(250)^2} = \frac{3.75\text{ ft.}}{(300)^2}$$

$$\text{tangent offset at 12+50} = (3.75\,ft)\left[\frac{(250)^2}{(300)^2}\right]$$

$$\text{tangent offset at 12+50} = 2.60\text{ ft.}$$

Sixth, calculate the elevation of the 12+50 on the tangent line

Elevation on station 12+50 on the tangent = (100 ft) + [+0.02 x 250 ft] = 105 ft

Finally, to find the elevation of the station the desired station on the vertical curve the tangent offset must be added or subtracted from the elevation of that station on the tangent, that is the straight grade line. The vertical curve in Figure 5 is a crest curve, therefore the relationship is:

Elev. of station on the vertical curve = Elev. of that station on the tangent – tangent offset

For a sag curve the relationship is:

Elev. of station on the vertical curve = Elev. of that station on the tangent + tangent offset

In this case it is appropriate to subtract the tangent offset for 12+50 from the elevation at the station on the tangent, the result is its elevation on the vertical curve:

Elev. of 12+50 on the vertical curve = 105 ft. – 2.60 ft.
Elev. of 12+50 on the vertical curve = 102.40 ft.

The answer is, station 12+50 has an elevation on the vertical curve of 102.40 ft

Method B

Example solution for a station on a symmetric vertical curve using a quadratic equation

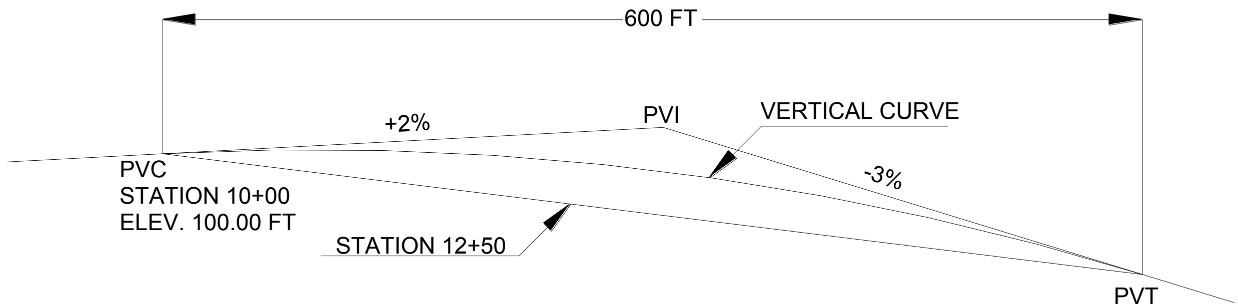

Figure 7

The general form of the equation is :

$$y = e + (g_1)x + \frac{r}{2}x^2$$

where:
e = the elevation of the beginning of the vertical curve , the PVC
g_1 = the first, or the entering, grade
r = the rate of change along the curve
x = the distance along the curve to the place for which the elevation is desired
y = the elevation of the place chosen

The objective is to find the elevation of station 12+50 on the vertical curve illustrated

First, find x, that is the distance along the curve to the place for which the elevation is desired:

Station of the desired elevation	=	12+50
Station of the PVC	=	-10+00
Length of x in stations	=	2+50
Length of x in feet	=	250 feet

Second, find r, that is the rate of change along the curve:

The value for the rate of change, r, is found by subtracting the first, or entering grade (g_1) from the second, or exiting grade (g_2) and dividing that difference by the total length of the vertical curve

The formula used is:

$$r = \frac{g_2 - g_1}{L}$$

where:
r = the rate of change
g_2 = the second, or the exiting, grade
g_1 = the first, or the entering, grade
L = the length of the vertical curve

Therefore, for the vertical curve illustrated above

$$r = \frac{g_2 - g_1}{L}$$

$$r = \frac{(-0.03)-(+0.02)}{600\,ft.}$$

$$r = \frac{(-0.05)}{600\,ft.}$$

$$r = -0.00008333$$

Third, find y.

Returning to the general form of the equation :

$$y = e + (g_1)x + \frac{r}{2}x^2$$

where:
e = the elevation of the beginning of the vertical curve , the PVC = 100.00 ft.
g_1 = the first, or the entering, grade = +0.02
r = the rate of change along the curve = -0.00008333
x = the distance along the curve to the place for which the elevation is desired = 250 ft.
y = the elevation of the place chosen = ?

Therefore:

$$y = 100\,ft + (+0.02)(250\,ft) + \frac{-0.00008333}{2}(250\,ft)^2$$

$$y = 100\,ft + \left(5\,ft\right) + (-0.000041665)(62{,}500\,ft)$$

$$y = 100\,ft + \left(5\,ft\right) + (-2.60\,ft)$$

$$y = 102.40\,ft$$

Now here is the same calculation done in stations rather than feet

$$r = \frac{g_2 - g_1}{L}$$

$$r = \frac{(-3)-(+2)}{6}$$

$$r = \frac{(-5)}{6}$$

$$r = -0.8333$$

Therefore, returning again to the general form of the equation :

$$y = e + (g_1)x + \frac{r}{2}x^2$$

where:
e = the elevation of the beginning of the vertical curve , the PVC = 100.00 ft.
g_1 = the first, or the entering, grade = +2
r = the rate of change along the curve = -0.8333
x = the distance along the curve to the statuin elevation desired = 2.5 Stations
y = the elevation of the place chosen = ?

$$y = 100\,ft + (+2)(2.5) + \frac{-0.8333}{2}(2.5)^2$$

$$y = 100\,ft + (5) + (-0.41665)(6.25)$$

$$y = 100\,ft + (5) + (-2.60)$$

$$y = 102.40\,ft$$

As you can see the calculation yields exactly the same result whether the units involved are feet or stations.

The answer is, station 12+50 has an elevation on the vertical curve of 102.40 ft

Asymmetrical Curves.

Sometimes an asymmetrical curve will fit found conditions more closely than the usual symmetrical equal-tangent curve. Suppose an engineer designed a sag symmetrical vertical curve for a route. The vertical curve as designed was intended to accommodate the rims of two existing manholes 800.00 apart. The entering grade was to be –4% and the exiting grade +3%. Unfortunately, the surveyor laying out the designed vertical curve finds that the two manholes are not 800.00 feet apart. The manholes are actually 872.43 feet apart. Nevertheless the both the entering grade and the exiting grade must remain as originally designed.

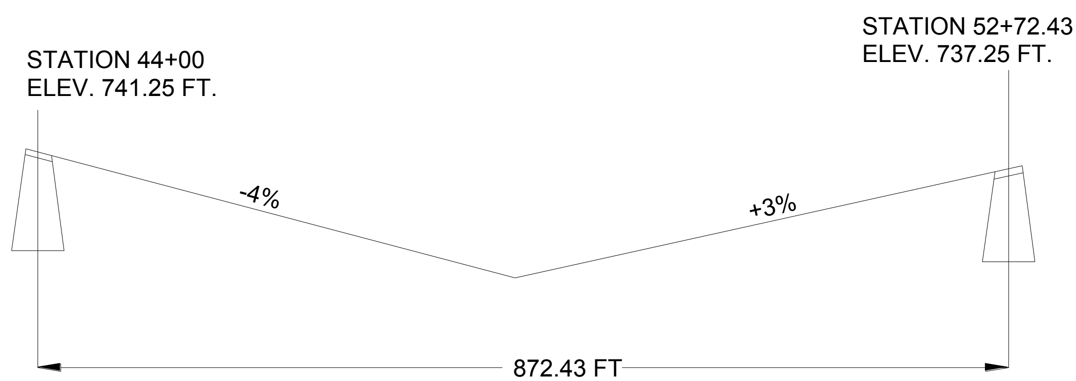

Figure 8

As you can see in Figure 8 the first manhole is at station 44+00 and its rim has an elevation of 741.25 feet. The second manhole is actually at station 52+72.43 and its rim has an elevation of 737.25 feet. The originally designed symmetrical vertical curve will not work since the distance is longer than expected and the grades must not change.

The first task will be to find the distance from the first manhole to the PVI, the intersection of the –4% and the +3% grade lines. Here is the relationship:

$$\text{Elev MH1} - g_1\, x = \text{Elev MH2} - (g_2)(\text{Distance MH1 to MH2} - x)$$

Where x is the distance from the first manhole to the PVI

Therefore:

$$741.25 \text{ ft} - 0.04x = 737.25 \text{ ft} - (0.03)(872.43 \text{ ft} - x)$$

$$741.25 \text{ ft} - 0.04x = 737.25 \text{ ft} - 26.17 \text{ ft} + 0.03x$$

$$0.03x + 0.04x = 741.25 \text{ ft} - 737.25 \text{ ft} + 26.17 \text{ ft}$$

$$0.07x = 30.17 \text{ ft}$$

$$x = 431 \text{ ft}$$

The distance from the first manhole to the PVI is 431 ft. Clearly then the length of the tangent lines for this vertical curve cannot be equal.

Distance MH1 to MH2	872.43 ft
Distance MH1 to PVI	-431.00 ft
Distance PVI to MH2	441.43 ft

An asymmetric vertical curve is required. The most straightforward method of constructing this sort of an asymmetrical vertical curve is to treat it as two equal tangent symmetrical vertical curves joined end to end. In other words, divide the first tangent length of 431.00 ft into halves of 215.50 feet each. Also, divide the second tangent length of 441.43 feet into halves of 220.715 feet each.

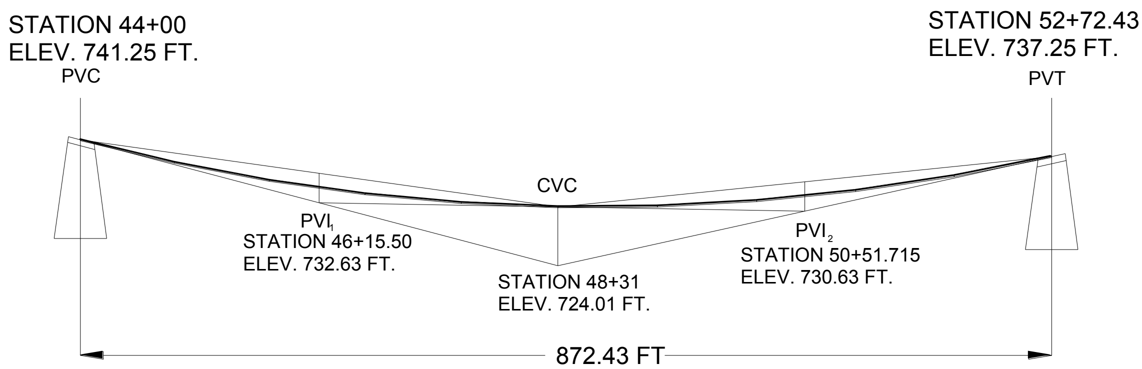

Figure 9

What remains then is to find the point of compound vertical curvature, *CVC.* As you see in Figure 9 this is conveniently done by connecting the PVI_1 and the PVI_2. The stations of these two PVI's are found in this way:

Station of the PVC	44+00.00
Distance from PVC to PVI_1 (in stations)	+<u>2+15.50</u>
Station of the PVI_1	46+15.50
Station of the PVT	52+72.43
Distance from PVT to PVI_2 (in stations)	-<u>2+20.715</u>
Station of the PVI_2	50+51.715

The elevations of these two PVI's are found in this way:

$$PVI_1 \text{ Elevation } = (\text{PVC Elevation}) - (g_1)(\text{Distance PVC to } PVI_1)$$
$$PVI_1 \text{ Elevation } = (741.25 \text{ ft}) - (-0.04)(215.50 \text{ ft})$$
$$PVI_1 \text{ Elevation } = 732.63 \text{ ft}$$

$$PVI_2 \text{ Elevation } = (\text{PVT Elevation}) - (g_2)(\text{Distance PVT to } PVI_2)$$
$$PVI_2 \text{ Elevation } = (737.25 \text{ ft}) - (+0.03)(220.715 \text{ ft})$$
$$PVI_2 \text{ Elevation} = 730.63 \text{ ft}$$

Please note that the station of the CVC is the same at the station of the point at which the straight grade lines, intersect, 48+31.00. Its elevation is found in this way:

$$\text{Grade from } PVI_1 \text{ to } PVI_2 = \underline{PVI_2 \text{ Elevation } - PVI_1 \text{ Elevation}}$$

Distance PVI_2 to PVI_1

$$\text{Grade from } PVI_1 \text{ to } PVI_2 = \frac{(730.63\text{ ft}) - (732.63\text{ ft})}{215.50\text{ ft} + 220.715\text{ ft}}$$

$$\text{Grade from } PVI_1 \text{ to } PVI_2 = \frac{-2.00}{436.215\text{ ft}}$$

$$\text{Grade from } PVI_1 \text{ to } PVI_2 = -.0046 = -0.46\%$$

Therefore, the elevation of the CVC is found this way

$$\text{CVC Elevation} = (PVI_1\text{ Elevation}) + (-0.0046)(\text{Distance } PVI_1 \text{ to CVC})$$

$$\text{CVC Elevation} = (732.63\text{ ft}) + (-0.0046)(215.50\text{ ft})$$

$$\text{CVC Elevation} = (732.63\text{ ft}) + (-0.9913\text{ ft})$$

$$\text{CVC Elevation} = 731.64\text{ ft}$$

It is now possible to find the elevation of any station on the first or the second portion of the asymmetric vertical curve using the the general form of the equation as illustrated above.

$$y = e + (g_1)x + \frac{r}{2}x^2$$

Here is a table of the elevations of each half station, that is, every 50 feet along the asymmetrical vertical curve illustrated in Figure 9

Description	Station	Elevation	1st Difference	2nd Difference (absolute value)
PVC	**44+00**	**741.25**		
	44+50	**739.35**	**-1.9**	
	45+00	**737.66**	**-1.7**	**0.2**
	45+50	**736.17**	**-1.5**	**0.2**
	46+00	**734.89**	**-1.3**	**0.2**
	46+50	**733.81**	**-1.1**	**0.2**
	47+00	**732.95**	**-0.9**	**0.2**
	47+50	**732.28**	**-0.7**	**0.2**
	48+00	**731.82**	**-0.5**	**0.2**

CVC	**48+31**	**731.64**		
	48+50	**731.57**	**-0.3**	**0.2**
	49+00	**731.51**	**-0.1**	**0.2**
	49+50	**731.65**	**+0.1**	**0.2**
	50+00	**731.98**	**+0.3**	**0.2**
	50+50	**732.51**	**+0.5**	**0.2**
	51+00	**733.24**	**+0.7**	**0.2**
	51+50	**734.16**	**+0.9**	**0.2**
	52+00	**735.28**	**+1.1**	**0.2**
	52+50	**736.59**	**+1.3**	**0.2**
PVT	**52+72.43**	**737.25**		

Table 1

In this table the consistent rate of change that makes vertical curves so attractive for grade transitions is illustrated by the *second difference.* In the column labeled as the 1st difference there is calculated the difference between successive elevations on even half stations, that is 50 ft. stations. For example, the difference, rounded to the nearest tenth of a foot, between the elevation 741.25 ft of the PVC at station 44+00 ft and the elevation 739.35 ft of the next station 44+50 is –1.9 ft . And the difference between the elevation 739.35 ft at station 44+50 and the elevation 737.66 of the next station 45+00 is –1.7 ft . Now the 2nd difference, that is the difference between –1.9 ft and –1.7 ft is 0.2 ft, absolute value. Please note that this second difference is consistent through the entire vertical curve. This illustrated that while the elevation difference between succesive station varies the rate of the change in the slope of the curve is constant.

Zero Slope - Turning Point on a Vertical Curve.

The turning point on the profile of a vertical curve, that is the point at which the slope is zero, is the curve's highest (or lowest) point. Please note on the curve above that this point occurs 49+00 and 49+50.

This point does not usually occur near the PVI, but is found on the right or left tangent. These high and low points are useful in analyzing drainage and overhead clearance problems. At the high or low point of a vertical curve a tangent line at that particular point would be horizontal. For example, in the crest vertical curve that was illustrated earlier

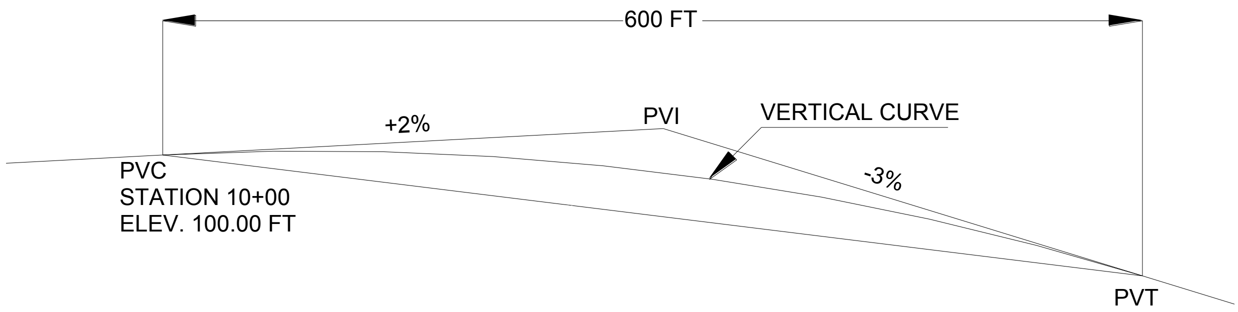

Figure 10

The formula that can be used to find high point of this curve from the PVC is:

$$x = -\frac{g_1}{r}$$

where:

x = the distance along the vertical curve to the high point
g_1 = the first, or the entering, grade
r = the rate of change along the curve

You may recall that the rate of change for this curve, in terms of stations, is -0.8333 and the entering grade is +2%.

Therefore:

$$x = -\frac{g_1}{r}$$

$$x = -\frac{(+2)}{(-0.8333)}$$

$$x = -(-2.4)$$

x = +2.4 stations = +240 feet

Since the station of the PVC is 10+00 the station of the high point of this curve is 12+40. This result can be checked by doing the same calculation from the PVT using the formula:

You may recall that the rate of change for this curve, in terms of stations, is -0.8333 and the exiting grade is
-3%.

$$x = -\frac{g_2}{r}$$

$$x = -\frac{(-3)}{(-0.8333)}$$

$$x = -(+3.6)$$

x = -3.6 stations = -360 feet

Since the station of the PVT is 16+00 the station of the high point of this curve is confirmed to be 12+40.

Typical Mistakes in the Calculation of Vertical Curves

1. Not checking the calculated elevations at the PVI, the PVC, and the PVT to be sure they match those that were given.
2. Calculating the tangent offset at the PVI incorrectly.
3. Adding tangent offsets in a crest vertical curve, or subtracting the tangent offsets in a sag curve.