PDHonline Course G361 (3 PDH)

Understanding Probability and Its Role in Decision Making

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2020

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List of Chapters and Subchapters:

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Introduction to Probability

The opening credits to the popular crime-solving TV series "NUMB3RS" include the statement that "We all use math every day; to predict weather, to tell time, to handle money. Math is more than formulas or equations; it's logic, its rationality, it's using your mind to solve the biggest mysteries we know."

Sometimes those numbers are concrete values, as in "The speed limit is 35" or "It is currently 68 degrees outside." But often, the numbers we see are not fixed, but rather are probabilities of future outcomes. We often use the phrase “I’ll take my chances.” But what does that really mean? Should our ‘gut feeling’ really guide our pick of lottery numbers? Or investment portfolios? Or career choices?

When most people think of probability, they think of games of chance: poker, roulette, etc. Much of the early mathematics of probability was developed for such games. However, we now realize that much of the world around us is probability -based. For example, the certainties of classical physics have given way to the probabilistic models of quantum mechanics and thermodynamics. In biology, we now understand genetics and evolution in a context of random behavior. These views represent major paradigm shifts. Closer to home, probability is pervasive in our everyday lives:

- Weather Forecasting
- Life and Homeowners Insurance
- Medicine and Consumer-Driven Care
- Finance and Retirement Planning
- TV Game Shows
- Sports
- Diet and Exercise
- Disaster Preparedness
- Uses of Technology
- Personal, Home and National Security
- Education
- Real Estate
- Foreign Travel
- Etc. etc. etc

Our lives are full of situations with unknown outcomes, whether we like it or not. While we usually cannot get rid of the uncertainty, what we can do is try to deal with the uncertainty in some sort of methodical or rational way.

Discrete Probability - Dealing with a Limited Number of Outcomes

Discrete Probability deals with situations that have a finite number of defined outcomes. For example, it will or will not rain today. My stock portfolio will go up, down, or stay the same.

In these cases, a probabilistic analysis of a situation provides a guide to help make decisions, but the results of the situation are still probabilistic. You can make a ‘good’ decision, and things might still turn out poorly.

For example, in the 1700s in Paris, there was a smallpox epidemic. A vaccine had been developed, but the inoculations were risky. One in every 200 people who were vaccinated died from the shot. However, the chance of dying from the disease itself was 1 in 7. Each person had to weigh the risk. Probability theory says that even if you died from the vaccine, getting immunized was still the correct choice. Not very comforting if you are the one that dies, but on average, many fewer people passed away because of the vaccine.

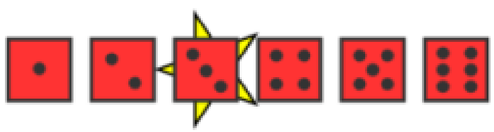
The Paris smallpox epidemic is a good example of Discrete Probability Theory. The theory cannot predict whether you personally will die from smallpox, but it can address the aggregate result over the larger population.

The Mathematics of Discrete Probability

In the broadest sense, discrete probability is the mathematics that provides a numerical measure of randomness. Here are a couple of discrete probability examples to illustrate the basic concepts. First, let us define an outcome.

Definition #1: OUTCOME - one possible result of a probability experiment.

For example, we say "rolling-a-3" is a possible outcome of pitching a die. Since there are 6 sides to a die, the probability of "rolling-a-3" is one out of six. That of course assumes the die doesn't end up precariously balanced on an edge, and also assumes that the die isn't weighted or magnetic. In this example, all outcomes have the same probability and the outcome of any experiment is random.



Rolling a Die has 6 possible outcomes, including rolling a "3"

Second, let us decide that we always want to express the probability of a particular outcome as a number between 0 and 1.

Definition #2: Zero means "never going to happen" and 1 means "definitely going to happen."

This is true for all 'disjoint' outcomes - that is outcomes that do not overlap. Using one die, you will either get a three, or you won't get a three. There is no middle ground.

Definition #3: Disjoint - outcomes of a probability experiment that have no overlap.

In the dice rolling experiment, we would say that the probability of "rolling a 3" is 1/6. We turn this around and ask "what is the probability of NOT rolling a 3?" Since there are 5 sides that are not a 3, the answer is 5/6. Let's add them together:

$$\text{Probability(rolling-a-3)} + \text{Probability(not-rolling-a-3)} = 1/6 + 5/6 = 1$$

Remember we defined "1" as "definitely going to happen". So what this formula says is when you roll the die, it's definite that you will either roll a 3 or you won't roll a 3. It's pretty hard to argue with that.

We can generalize this as Axiom #1:

Axiom #1: The sum of the probabilities for all possible disjoint outcomes of an experiment equals 1.

We have several ways to express probability: as a fraction, a decimal, or a percentage. We typically use the fractional form when we are thinking about the specific number of outcomes and the chance that our desired outcome will happen. We use the percentage more to mean, if I did this experiment 100 times, how many times could I expect that my desired outcome would occur?

So the probability of rolling-a-3 could be expressed as 1/6, or 0.1667 or 16.67%. Of course it is easy to switch between the forms, as they are all equivalent.

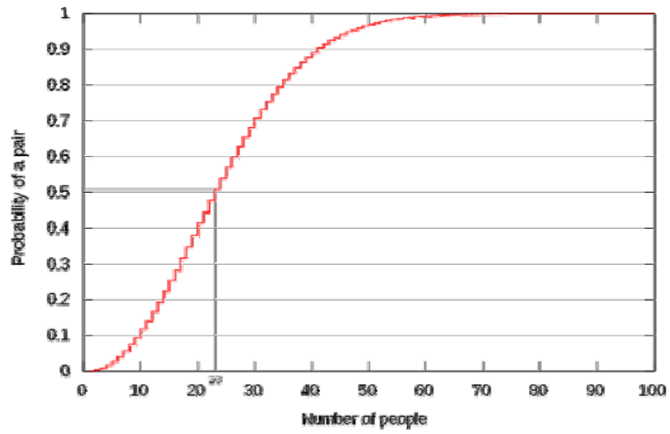
The power of Axiom #1 is illustrated in the following example.

The Birthday Problem

You go into room with 50 people. What is chance that two people in the room have the same birthday (month and day)? Most people guess that this is pretty rare, but it actually happens 97% of the time!

Let's see why. There are two disjoint possibilities: either at least two people have the same birthday, or everyone has a different birthday.

Because these are disjoint, their probabilities sum to 1 (Axiom #1). So we choose the easier to calculate, which in this case is the second one. So we poll people, one by one. We ask person 1 what her birthday is. Then, what is the probability that person 2's birthday is different? The answer is: 365/366 because there are 365 days left to choose from. Move on to person 3. The probability that his birthday is different than the first two's is 364/366. We can continue thus until we get to person 50, whose birthday is different than the other 49 with probability 317/366. Then the cumulative probability is the product of all of these, or 365/366 * 364/366 * ... 317/366 = .03, or 3%. This means that there is a 3% chance that nobody has the same birthday as anyone else. By Axiom #1, this also means that there is a 97% chance that at least two people have the same birthday. We can repeat this calculation for different numbers of people in the room. For example, there is a 50% chance of a shared birthday with 23 people in the room, and with 90 people it jumps to 99.9993%. The graph below shows the distribution.



The probability of two identical birthdays in a random group of people.
(This graph is from http://en.wikipedia.org/wiki/Birthday_problem)

Let's look at some groups of people to see if this holds true experimentally:

- Birthdays of U.S. presidents (of which there are 44): Warren Harding (29th) and James Polk (11th) both were born on November 2.

- Death dates of U.S. presidents: Millard Fillmore (13th) and William Howard Taft (27th) both died on March 8, and John Adams (2nd), Thomas Jefferson (3rd) and James Monroe (5th) all died on July 4.

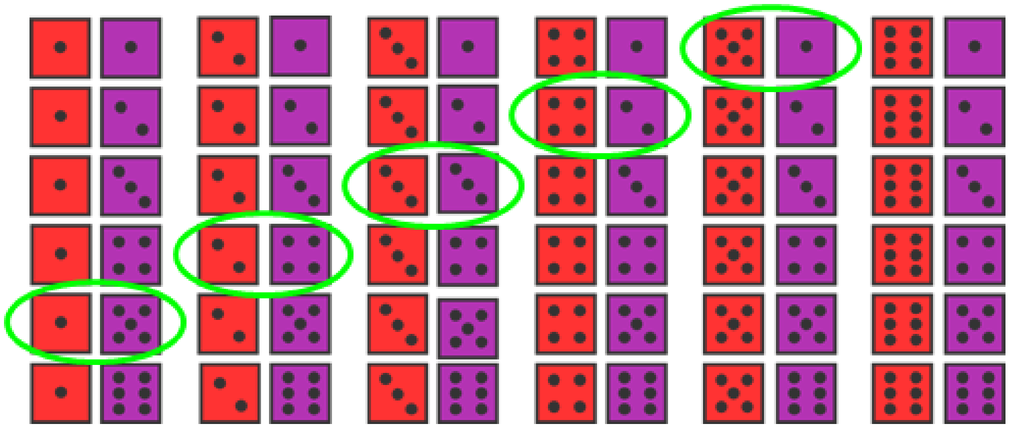
- Birthdays of Best Actress Oscar Winners from 1950-1999(of which there were 43): Joanne Woodward (1957's The Three Faces of Eve) and Elizabeth Taylor (1960's Butterfield and 1966's who’s Afraid of Virginia Woolf) were both born on February 27.

- Birthdays of Best Actor Oscar Winners from 1950-1999 (of which there were 47): both Sir Ben Kingsley (1982's Gandhi) and Anthony Hopkins (1991's The Silence of the Lambs) were born on December 31.

Combinations and Permutations

Say we have two dice, one red and one purple, and we roll them both and sum the values on the faces. What is the probability that the sum of the numbers is 6? The possible values that sum to 6 are 1 and 5, 2 and 4, and 3 and 3. Recognize that there are two die, so red 5 and a purple 1 is a different outcome than red 1 and purple 5, even if the sum is 6 either way.

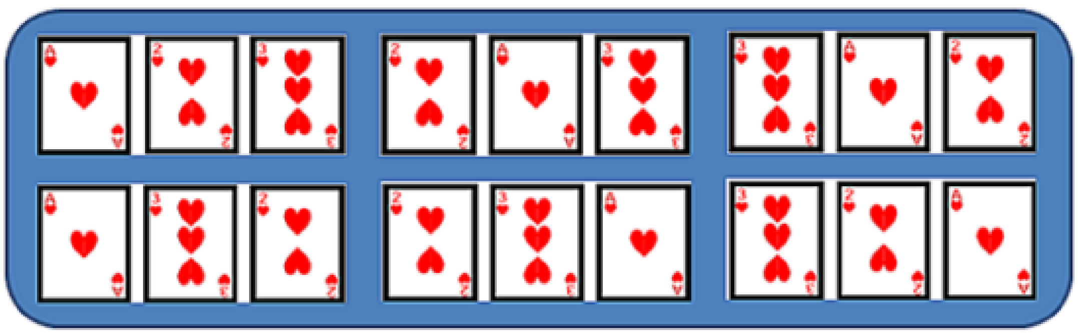
So there are actually 5 ways to get a sum of 6: {1+5, 2+4, 3+3, 4+2, and 5+1}. There are 6 choices for the red die, and for each of those there are 6 choices for the purple die, so there are a total of 36 combinations of faces. So the probability of the sum being 6 is 5/36 or about 14%. This is graphically illustrated as so:



Rolling Two Dice -36 possible outcomes. Five of those outcomes sum to 6.

We can generalize this to say that finding the probability of an event just comes down to a question of counting. As it turns out, this can be a lot harder than you might expect. There is a whole field of mathematics, called combinatorics, which deals with counting problems.

Let's address some basic counting questions by example. If I have 3 cards, how many different ways can I arrange them in a row? The answer is 6, as shown here:



Six different ways to order the same three cards.

So in this case we say there are 6 PERMUTATIONS of three cards. Therefore, we define:

Definition #4: PERMUTATION - An-arrangement of a set of items.

How do I calculate the number of permutations? Think of it this way. You have 3 choices for which card comes first. You then have 2 choices for which one comes second. Then there is only 1 choice for the third. The number of permutations is calculated by multiplying the numbers of choices together:

3 times 2 times 1 = 6

We do this so often in probability, that we have special name for it - a **FACTORIAL**, which we indicate with an exclamation point (because we think factorials are so cool!)

Definition #5: FACTORIAL The result of multiplying a series of descending positive integers. For example, 4! is 4 * 3 * 2 * 1 = 24

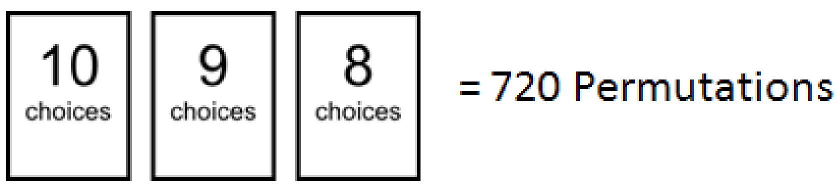
So, the number of permutations of 3 cards is 3-factorial, or 3! which means: 3*2*1 =6.

How many ways are there to order 5 cards? That should be 5! = 5 * 4 * 3 * 2 * 1, which is 120. Try it if you don't believe me. (It will be on the quiz).

Factorials increase very rapidly with the number of cards. For example, the number of permutations for 10 cards is 10!, or 3,628,800. The number of different ways to order the entire deck of 52 cards in a row is 52!, which is about 8 followed by 67 zeros. No wonder we use the exclamation point.

Ok, let's make this a little more complicated. Take a subset of a deck of cards, composed of ace through 10 of hearts, with no face cards (10 total cards).

How many different ways are there to pick three of them? That is, how many permutations are there of 3 cards out of 10? Well, there are 10 choices for the first card, 9 choices for the second card, and 8 choices for the third card. So based on our permutation discussion above, there are 10 x9 x8 =720 possible ways.



Calculating the Number of Permutations for 3 cards picked from 10 cards.

We can generalize this to a formula:

Formula #1: Number of Permutations of K objects from N objects = $\frac{N!}{(N-K)!}$

So in our case, N=10 and K=3. Do the math and you do, in fact, get 720.

Note that in this example, picking a 2, 3 and 5 is one permutation, picking a 5, 2 and 3 is another, 2, 5 and 3 is another, as is 3, 2, 5, etc. But if I'm playing cards, these are all the <u>same hand</u> - one hand of 2, 3 and 5. We know from our first example that there are 6 permutations for 3 cards. Therefore, the actual number of hands with 2, 3 and 5 in <u>any</u> order is 720/6 or 120. This leads us to the definition of a COMBINATION:

Definition #6: COMBINATION - A set of items, in which the order does not matter.

We can write a formula for how many ways to pick 'K' items from 'N' total objects, <u>when order doesn't matter</u> (like 3 cards from 10 cards in the example). Let's call it "From N choose K" and express it as so:

Formula #2: Number of Combinations of K objects from N objects = $\frac{N!}{(N-K)! * K!}$

There will always be fewer combinations than permutations. For example, all the arrangements of the three cards we discussed about yielded six permutations. But these six permutations are one combination, since a combination doesn't care what order they are in.

Since factorial math can get hairy, here is the address of a website that calculates both the number of permutations and combinations for any From N choose K problem. No, you won't need it for the quiz; it's just fun to experiment with.

http://www.calctool.org/CALC/math/probability/combinations

Keeping Things Straight

It can be confusing to remember the difference between combinations and permutations. Here is the trick I use.

What is wrong with this phrase? : "The combination to the safe is 23-21-86."

By definition, in a combination the order does not matter. In opening a safe, the order of the numbers certainly does matter! So, mathematically speaking, the correct phrase would be: "The permutation to the safe is 23-21-86."

Examples of Combinations in Straight Poker

In the original game of straight poker, you are dealt 5 cards from a standard 52-card deck. What is the probability of drawing the following hand? (Remember, order doesn't matter when talking about 'hands'.)

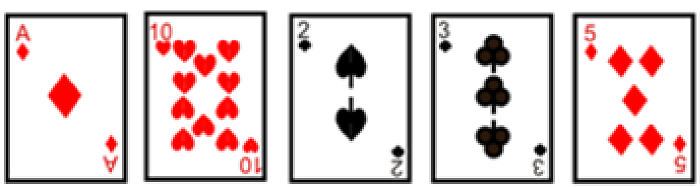


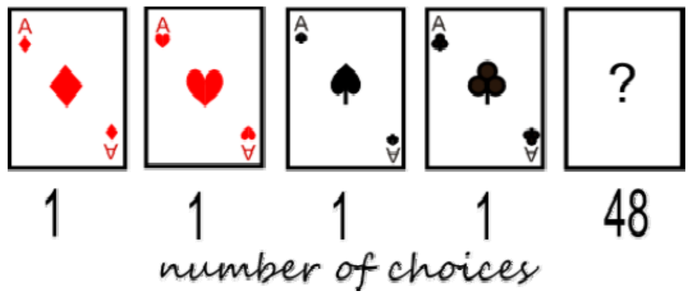
Figure 6 -One Possible Hand in Straight Poker

For this we would use our Combination Formula. Using the formula for choosing 5 cards out of a deck of 52 gives us 2,598,960 possible combinations , most of which are losers, unless, of course, you are very good at bluffing.

So the probability of being dealt this particular hand is 1/2,598,960 or about 1 in 2.6 million.

Note that EVERY hand has an equal chance of being dealt. So the hand shown has the same probability of being dealt as a hand with 4 aces and the king of hearts. However, you are much more likely to remember the latter (We will return to this idea later.)

What is the probability of being dealt four aces? We just count the number of outcomes that satisfy this requirement. Four of the cards are specified (the aces), so we just need to see how many choices there are for the 5th card. There were 52 cards in the deck, but we already got the 4 aces, so there are 48 cards left. So there are 48 different hands that contain all 4 aces, one for each choice of a last card.



A total of 48 different hands have all four aces out of a possible 2,598,960 hands says that the probability of getting a hand with all 4 aces is 48/2,598,960 or roughly 1 in 50,000. Feeling lucky yet?

Let's look at a couple more examples:

- A flush is any five cards in the same suit. There are 5,108 such combinations, so the probability is about 1 in 500.

- A straight is five sequential cards in any combination of suits. There are 10,200 such hands, so the probability is about 1 in 250.

This is how the ranking of the different hands is determined. You are twice as likely to get a straight as a flush. The rarer hand is of higher value, so a flush beats a straight.

The Law Of Large Numbers

Looking at a random process in the short term may not lead to any logical conclusions, but if we do it enough times, we can start to see the pattern. We may not be able to predict any individual result, but we can predict the long-term 'average' behavior. For example, say I roll 100 ordinary dice. There is an equal chance that I will roll a 1, or a 2, or a 3, up to 6 on each die. I can expect that I will get about the same number of 1's, 2's, and so on. That is, I expect that on average, I will get about 17 (1/6 = 16.66667%) rolls showing each of the numbers 1 to 6. In practice, I may get more or fewer for any one number, but in all likelihood it will be fairly even across the possible values. This is codified in the following statement.

Definition #7: THE LAW OF LARGE NUMBERS (LoLN): If we conduct an experiment which has many different possible outcomes and repeat the experiment many times, the fraction of trials that exhibit our desired event divided by the total number of trials will approach the mathematical probability of that event.

In his lecture series "What Are the Chances? Probability Made Clear," Dr. Michael Starbird has a computer simulation that rolls a virtual die many times. Consider the following chart:

Number of Rolls	Number of 3's	Percentage of 3's
6	0	0%
60	9	15%
600	92	15.33%
6,000	997	16.62%
60,000	10,037	16.73%

As he runs the simulation more times, the number of 3's divided by the total number of trials gets closer to our calculated probability of rolling a 3, or 1/6=16.67%. That is, the actual outcome gets closer to the expected outcome the more times we run the experiment.

Let's wrap up our discussion here with an example of the Law of Large Numbers in action. In 1929, The English astronomer Sir Arthur Eddington stated, "If an army of monkeys were strumming on typewriters, they might write all the books in the British Museum." The LoLN tells us that given enough time, the monkeys randomly hitting keys would certainly hit the right keys in a row to do so.

To get a grasp of the enormity of this task, however, let us consider a much smaller example. Hamlet said: "To be or not to be." Assume there are just 26 keys on a typewriter. There are many possible strings of 18 characters, only one of which is the desired one. In this case we do care what order the letters are struck, so we are interested in the number of Permutations. Using the Permutations formula for 'From 26 choose 18' we get 1 followed by 22 zeros.

How big is that number? Say one monkey takes 3 seconds to type 18 characters. There are 30 million seconds in year, so this monkey types 10 million phrases a year. Let's use 1,000 monkeys. If they all started on the day the earth was born, about 4.5 billion years ago, there is still only a one in a billion chance that one of them would have typed "To be or not to be" by today. So they just need some more time.

The Psychology of Rarity

It is worthwhile to take a moment to consider what it means for an event to be "rare," and our psychological interpretation of the concept of rarity. Say I'm going to play the lottery. I get to pick 6 numbers from 1 to 100, and if they all match the ones chosen by the state, then I win. I usually play the numbers 1, 2, 3, 4, 5, 6. Most people think I'm crazy. "What are the chances of that?" they gasp. They think somehow I'm picking a series that would be unbelievably rare and I could never win.

Actually the chance of the winning numbers being mine is exactly the same as the chance of any other set of 6 numbers, including your kid's birthdays, your lucky number, or a random number picked by the computer. (What is that chance? Since I don't care what order they are picked, I use the Combination formula. The number of possible combinations of 6 numbers is "From 100 choose 6", or 1.2 billion.)

Here is another example. Say you are concerned about your life, and you go to see a psychic reader. She will make many (sufficiently vague) predictions about your life. If she makes enough of them, there is a good chance that at least one of them will come true (and of course a better chance now that you are looking for them). She is taking advantage of the LoLN, that if each prediction only has a 10% chance of being correct, she just has to make 10 of them, and most of the time one will be right. If just one comes even slightly true, you are likely to remember how spot on her prediction was, and likely to forget all the ones she said that were not. Then you may become a regular customer, caught by the Psychology of Rarity.

The Psychology of Randomness

This is a fascinating experiment. Ask a friend to imagine flipping a coin 50 times and recording whether it landed heads or tails. A typical outcome of this mind-flip exercise looks something like this:

HHTHTHTTHTHTHTTHTHTHHTTHTTHTHHHTTTHHTTTHTHTHHTTTHT

Have your friend write it down but not show you.

Now ask your friend to actually flip a coin 50 times and record the result. Here is a typical run:

HHHHTHTHTTHTHTHTTHHHTTTTTTTHTHTTHHHHHTHTTHTTTHHHHH

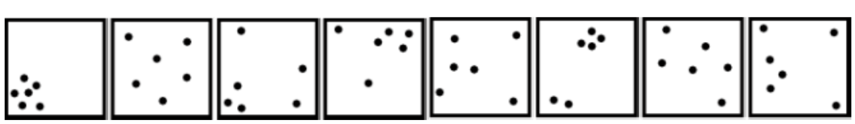
Now looking at both the imagined and real run, you can tell immediately which is the real run. How? Look for long strings of the same outcome. Why, you say? For 200 coin flips in a row, the probability of getting at least one string of 6 or longer is 96%. The probability of getting a string of at least 5 in a row is 99.9%. But my friend did not have any strings longer than 3, because we humans tend to think that long strings are not random.

Now suppose that you flip the coin 10 times and it comes up heads each time. Don't you feel like the coin 'wants' to make a tails next time? Rationally you know that there is no reason that the coin would care: there still a 50% chance of heads. We could formalize this as follows:

Axiom #2: For independent events, previous outcomes do not influence future probabilities.

If you flip a coin 11 times, there is only a 1/1024 chance that your first 10 flips will be all heads. So if this happens, you might feel that it is extremely unlikely that you would get another heads. However, the last flip still has a 50/50 chance of being heads. For most people, the psychology of randomness trumps rational thought.

Let's consider a visual example of how our brains perceive randomness. Here are 8 squares with six dots drawn in each square:



Ask yourself, or your friend, which of these is the "most" random? Most will pick square #2. Alternately, draw a blank square and ask your friend to draw 6 dots at random. Most of the time you will get a distribution that looks a lot like square #2.

Only a geeky mathematician would draw something like square 1, but the fact is all eight squares are equally random.

The mistake we make in our minds for both the coin-flip and the six-dots-in-a-square is that we equate “random” with “equally distributed”, two concepts which are definitely not equivalent.

The human brain is programmed to look for patterns, to make order out of the chaos of life. A great example of this is the distribution of the stars in the night sky, which is completely random. And yet for centuries people have tried to make sense of them. We search for groupings of bright stars and look for patterns within those, thus the constellations! We make order from chaos.

Expected Values

So far we have considered the chances of events happening, but we haven't yet considered the implications of any particular event. Yes, the chance of being dealt a hand with all four aces plus the king of hearts may be the same as any other hand in poker, yet in this case you may have won a lot of money, making that hand worth more than most others. In poker, we can assign a value to every possible hand, including zero.

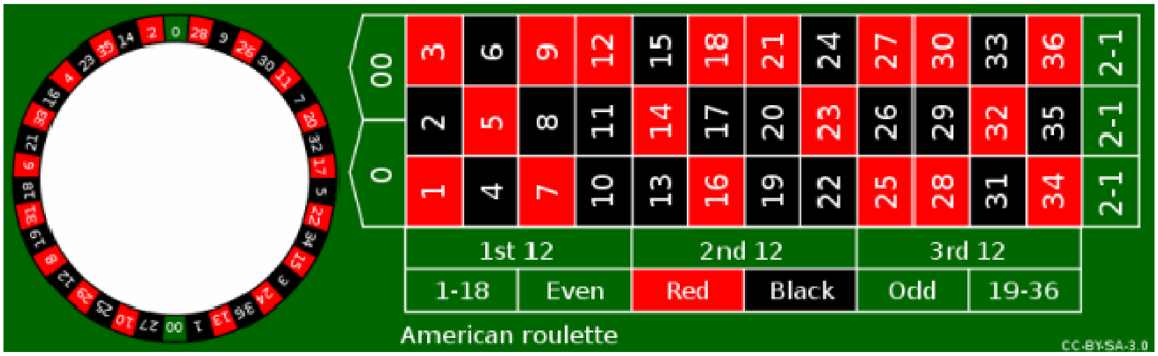
We have to make decisions in our lives by weighing the possible outcomes and their repercussions (or value). The classic example is buying insurance. Is it worth paying insurance for my house, or am I willing to risk the chance that everything will be fine, in order to save money? If there is a hurricane, then I'm going to regret not having bought it. But what are the chances of a hurricane destroying my house? Is it worth paying for the insurance so that I don't have to worry about it?

We have to make choices even though we cannot get rid of the uncertainty. But, the cost of the insurance will certainly play a role in this decision, as will the payout in the event of a hurricane. We need a mathematical model that takes into account both the probabilities and the value to give us a way to analyze the problem in spite of the uncertainty.

Definition#8: EXPECTED VALUE - The weighted average of the value of all possible outcomes.

We note that the expected value is not necessarily one of the possible values of the outcomes, but rather a weighted average over many repetitions of the experiment that gives us an indication of what our chances might be on any particular run.

The Game of Roulette



Let's consider the game of roulette. We have a wheel with slots labeled with the numbers 1 to 36, plus 0 and 00, and we drop a ball into this wheel after it is spun. The probability of the ball landing in any given slot is 1/38. First let us consider betting on a single number and try to reason out what kind of outcomes we could expect. The single-number payoff for winning is 35 times your bet. So, if I bet $10 on 17, then if the ball lands on 17, I win $350, and I get my $10 back. If not, I lose my $10. The probability of the ball landing on 17 is 1/38. Say I spin the wheel 38 times, and one time I win. Then I will have won $350 one time, but lost $10, 37 times. I have a net loss of $350-10($37) = -$20, over a total of 38 games. So, on average, I lost $20/38 = $0.53 per game.

If I play $10, the expected value of matching a single number is roulette is $ -0.53.

(The expected value varies with bet. For $1.00 bets, the expected value is $ -0.053, and for $100 bets, the expected value is $ -5.30, but the ratio is the same, and always a minus number.)

We have two possible outcomes: the ball lands on 17 and we win $350, or the ball does not land on 17 and we lose $10. So using our now familiar probability math:

$$\text{Expected Value} = (1/38)(\$350) + (37/38)(-\$10) = -0.53$$

Let's consider another bet in roulette. Half the numbers are red, and half are black, while 0 and 00 are neither. We can bet that a specific color will come up instead of a specific number. The payout for a single color is 1 to 1. So a $10 bet can win you $10.

What is the expected value if I play $10 on red? The probability of getting red is 18/38 and the probability of not getting red is 20/38. So we just use our formula and get:

$$\text{Expected Value} = (18/38)(\$10) + (20/38)(-\$10) = -0.53$$

Note that this is the same minus-53 cents as the previous case! Over the long term, betting on a color gives the same payout (or rather, loss) as betting on a single number. In fact, there are 22 different ways you can bet in roulette and lots of different payouts. For $10 bets, 21 of those ways all have an expected value of minus 53 cents. And the 22nd way has an expected value of minus 79 cents.

Based on the LoLN, if we play long enough and place $10 bets, we will eventually have lost an average of 53 cents a game. The LoLN favors the casino. But people keep playing roulette, and lots of people make money. So what are we missing here?

We can apply the same math we have been doing all along to see what is happening on any given evening at a roulette table, all based on the Expected Value Theory.

To keep it simple, say you are at a roulette table with a bunch of different players, and everybody plays 35 rounds, each betting $10 on one number of their choosing each time. Then the expected value of those rounds is 35($ -0.53) = -18.42 per person. Based on this logic, everyone should lose about $18.

However, probability says that everyone losing would be extremely rare. In order to be ahead at any given time in those 35 rounds, a player would only have to have won just one time. What are the chances of that? The probability of winning at least once is 1 minus the probability of losing all 35 games.

This turns out to be 1- (37/38)^35 = 0.6 (where ^35 means 'raised to the 35^{th} power').

Thus, 60% of the people at the table will be ahead at any given time. The important thing to note here is that people who are behind will have lost more than the people who are ahead have won. The Expected Value holds true because the losers lose more than the winners win. But if you are a winner, who cares?

Expected Value and Disappointment

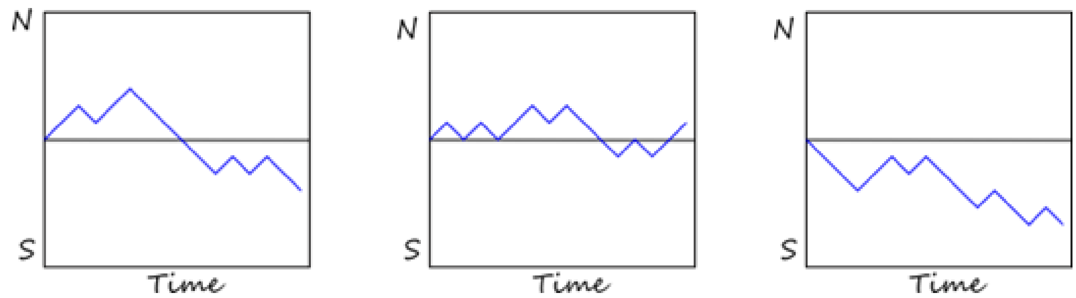
We have to be careful with the Expected Value, because it doesn't always mean what we intuitively think it should. Say I go to the DMV to renew my license. They know from experience that the wait time generally varies from 1 to 11 minutes. So they tell me that the average wait time is 6 minutes. (1+11)/2=6. However, the Expected Value of the wait time is actually 11 minutes. Let's see why that is.

Say I have a bucket full of strings. Half are of length 1 inch and half are of length 11 inches. Then the average length of a piece of string is (1+11)/2 = 6". First notice that although the average piece of string is 6" long, there are no 6"-long strings in the bucket. Also notice that when I reach into the bucket and pull out a piece of string, I am 11 times more likely to pick a long string than a short one, simply by virtue of its length. There is simply 11 times more string available for me to pick up.

In the case of waiting in line at the DMV, you are much more likely to be in one of the long waits, simply because they are longer. I have personally noticed that at crowded restaurants, they always tell me it will be about a 15-minute wait. OK, I say. Oh, how my expectations can quickly turn to disappointments, even though I fully understand the math behind Expected Values!

Random Walks

Say I am going to take a walk on a North-South street. I have no particular goal but to get some exercise, so I want to make it more interesting than just picking a direction and sticking with it. I have a coin in my pocket which I flip at the end of each block. If it comes up heads, I will go north one block. If tails, I will go south one block. Here are a few graphs that shows possible walks, where the x-axis represents time.



So what can we figure out about this walk? Where will I end up? In probability theory, this is called a "Random Walk".

Definition #9: A Random Walk is a series of steps, where the direction of each step is both random and independent of the previous step.

So now, what is the probability that at some point in the future, I will end up back at my house?

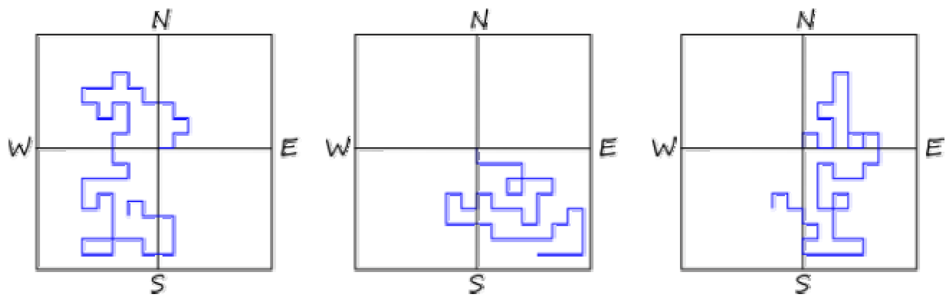
Let's start with the simplest case: say I am one block north of my house. What is my chance of getting home in one step? My movement is determined by the flip of the coin, so there is 1/2 chance I'll go north, and 1/2 chance I'll go south. That is, a 1/2 chance that I get home in one step, and a 1/2 chance that I will have to try again from 2 blocks away.

Let P represent the probability of getting home in one step, from 1 block away. We have P = 1/2 + 1/2 Q where Q is the probability that, starting 2 blocks away, I will eventually get back to 0. In order to get from block 2 to my house, I'll have to pass through block 1, which happens with probability P. (This is called recursion, where apply a method multiple times.) Then once I'm at block 1, there is probability P that I get back to my house from block 1. That is, Q = P*P, or P^2. We substitute and do a little algebra (don't worry if this is too hard):

$$P = 1/2 + 1/2\, P^2$$
$$0 = 1/2 + 1/2\, P^2 - P$$
$$0 = P^2 - 2P + 1$$
$$0 = (P-1)^2$$
$$0 = P-1$$
$$P = 1$$

That means I will get home with a probability of 1 (eventually). But wait, you say, what about the possibility that I flip heads every time? Then I will never get home. Since we didn't put a time limit on our adventure, we could be walking for infinitely long. Then there are infinitely many possibilities, of which just a few (like all heads) will never get us home. The probability of getting one of those is so small that, in practice, we will always get home, eventually.

Just for comparison, what happens if we do a random walk in two directions, where at each corner we randomly choose to go north, south, east, or west? We can ask the same question: what are the chances that I will end up back where I started? The probability is in fact still 1, but the rate is not as quick as it was before. That is, it is much more likely that it will take me many more steps to get back, but I will eventually get there.



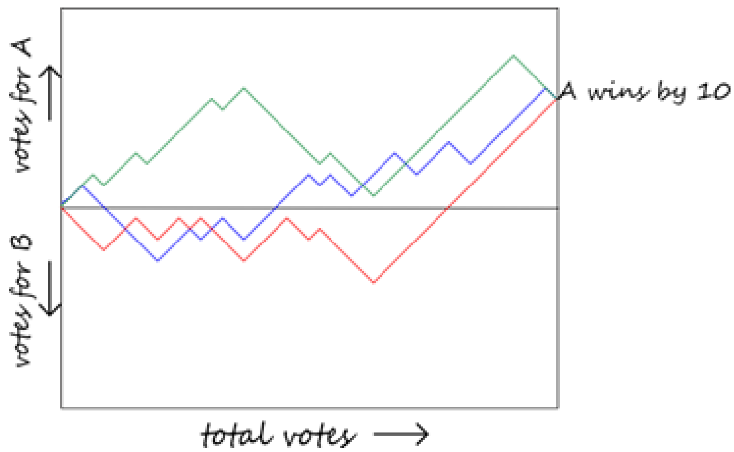
The return to home is a fundamental concept in Random Walk Theory. This leads us to a classic example in gambling called Gamblers Ruin.

The Gamblers Ruin

Say you start with $10 at a casino with completely fair odds, and a game that consists just of flipping a coin for a $1 bet. If you get heads, you win $1; if you get tails, you lose your bet. We can think of your take as a random walk, where each step is either $1 up or $1 down. Our previous example proves that if you continue to play indefinitely, you have a 100% probability of going broke. This is what's known as the GAMBLER'S RUIN. It may only take 10 bets, or it may take more than 1000, but eventually you will lose it all. As the classic Kenny Roger's song "The Gambler" says: "You've got to know when to hold 'em, know when to fold 'em, know when to walk away, and know when to run."

The Bertrand Ballot Theorem

Here is another random walk example. This one entails counting votes out of a ballot box. For simplicity, assume there are two candidates and only 51 ballots. One person will win, only you don't know who yet. What are the chances that as you count the ballots, one by one, the winner will always be ahead? We can graph this as the difference in the votes, where we go up one unit if A gets the vote, and down one unit if B gets the vote. Then when the graph is above the axis, A is winning, and when it is below, B is winning.



Notice in this example that if we count them in the green order, A is always winning. In the blue order, A is winning at the start, and then B takes over for a while, then A takes the lead again. In the red order, B is actually winning for most of the count, but then A pulls ahead at the end. The total vote count is the same, but it just depends in which order we count the votes. This issue was first addressed by Joseph Louis François Bertrand in 1887.

BERTRAND BALLOT THEOREM: Let candidate A get a total of a votes, and candidate B get a total of b votes, where a>b.
Then the probability that A stays ahead the whole time is = (a-b)/(a+b).

So if you are watching the presidential election results coming in slowly, keep in mind that just because one candidate is ahead at any given time, doesn't necessarily mean that she will be ahead overall.

Other Applications of Discrete Probability

Science and mathematics is continually advancing with theories being tested, rejected, revised or replaced. Over past 100 years, probability has taken a larger and larger role in our modeling of the world around us. The following sections describe some of the paradigm shifts this has forced.

Probability in Physics - The Nature of Matter

Some of the earliest theories on matter postulated that matter was made of discrete particles - atoms and molecules. At the turn of the 20th century however, physicists were not so sure that such things actually existed. They were thought of as metaphorical concepts or abstractions that provided useful predictive models, but did not necessarily reflect truth. The first evidence for their actual existence was in fact an application of probability.

In 1827, Robert Brown, a Scottish botanist, made microscopic observations of grains of pollen in water. He saw that they were constantly moving in a random, jittery way, without slowing down or stopping. He began to characterize their motion in a mathematical way, now called **Brownian Motion**. In 1905, Einstein hypothesized that Brownian Motion was caused by atoms and molecules hitting the grains of pollen, causing them to define a random walk. He developed a formula that gave a quantitative prediction of how much a piece of pollen would move per unit time, based on the size of the grains of pollen, the temperature and the viscosity of the liquid.

Einstein's paper on Brownian Motion was just one of three significant papers that he published in 1905; another was on the photoelectric effect and the final on relativity. In 1908, Einstein's results were verified experimentally, providing evidence of the existence of atomic particles, but at the same time, introducing the concepts of probability and randomness to what was previously thought to be an exact science.

This marked the beginning of a fundamental shift in the type of analysis used for physical quantities. For example, in quantum mechanics, the fundamental particles are not viewed as discrete points in space and time, but rather as a probability distribution that tells you the probability that the particle exists at any given point. You can think of this as a particle (such as an electron orbiting a nucleus) existing simultaneously in multiple places, with different probabilities for each location, and it is only when you measure its location that it is forced to pick one. If you measure it many times, then it will fill out that probability curve (remember the LoLN?). This approach represents a real paradigm shift, one that Einstein did not even accept in his day. He famously stated, "God does not play dice with the universe." However, physicists today are in agreement that this is the best model we have.

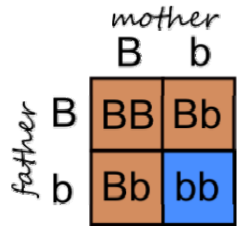
The famous thought experiment called "Schroedinger's Cat" was devised by Erwin Schroedinger in 1935, to study the problem of quantum mechanics applied to objects not on a quantum scale. We consider a closed box that contains a cat, a radioactive source, and a Geiger counter (which measures radiation) connected to a flask of poison. If the Geiger counter detects radiation, then the flask will open and the cat will die. Say there is a 50% chance of a radioactive particle hitting the detector. Then, there is a 50% chance that the cat is alive. Quantum mechanics states that the cat exists in a state that is simultaneously alive and dead, and it's only when you open the box to check that it must pick one.

Probability in Genetics - Eye Color Today and In The Future

The study of genetics deals with how the characteristics of parents are passed on to their offspring, based on the idea that the genetic material of the parents randomly combines in their children. They can get different traits depending on how their parents' genes interact. Thus, the concept of probability plays a central role in predicting how the offspring will come out.

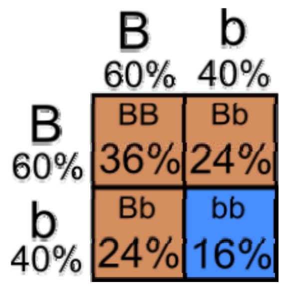
For our purposes, we are going to simplify the model of genetics. Assume that eye color is determined by only one gene (biologists know this isn't true) and that you can only have brown (B) or blue (b) eyes (everyone knows this isn't true). Each person carries two ALLELES in his/her gene. Brown is a dominant allele and blue is a recessive allele, meaning that if brown is present, brown will be expressed over blue. Blue only appears as eye color if both alleles are blue. That is, you can have BB (brown), Bb (brown), or bb (blue).

Say we have two parents, both of whom carry Bb. Then the following chart shows us the possible outcomes for an offspring:



The offspring only has a 1/4 chance of having blue eyes, and a 3/4 chance of having brown eyes.

Now let's consider a whole population. Say that 60% of the alleles are B and 40% are b. We randomly select two parents from this group and investigate their offspring. What percentage of their offspring will have brown versus blue eyes? What about the alleles?



Doing the math says that 16% of the next generation will have blue eyes. You might guess that if an allele is recessive, over time it would "die out." When you do the mathematics, however, this is not the case. Even in the case of recessive characteristics like blue eyes, there is a stable percentage of those alleles in the population, assuming a stable rate of procreation. This concept was independently discovered by an English mathematician, Godfrey Harold “G. H.” Hardy (1877 – 1947) and Dr Wilhelm Weinberg (1862 – 1937) a German physician. It is now referred as:

HARDY-WEINBERG EQULIBRUIM THEOREM: There will be the same proportion of alleles in the population from generation to generation, unless specific disturbances are introduced.

In practice, however, there are always disturbances: non-random mating, mutations, natural selection, genetic drift, inter-breeding and population size limitations, to name a few. In fact, the proportions of alleles in a population actually behave more like a random walk. It's possible that over time there will be more and more blue eyed babies, until everyone has blue eyes. Or, over time they might die out completely.

Probability in Medical Screening

Let's look at another example where the reality may not agree with our intuition. Consider the test to determine if a person has HIV. For people who don’t have HIV, the test returns a false positive 1% of the time. For people who have HIV, the test comes back positive 95% of the time.

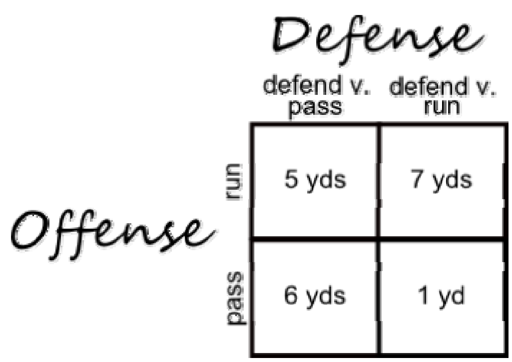
A patient takes the test and gets a positive result. What is the chance that that patient has HIV? Assume that that in a particular city, 300,000 people are getting tested and 500 of them actually have HIV. Then, 95% of those with HIV will test positive (475). The other 299,500 people do not have HIV, but 1% of them will get a false positive (2,995 people). That means that 3,470 people get positive results, but only 475 of them actually have HIV.

So in the case of our patient who got a positive test result, the chance of actually having the disease is only 475/3470 = 15%.

Game Theory and the Nash Equilibrium

Game theory is a branch of applied probability theory in which we try to analyze people's behavior in certain strategic situations (called games), in which any one person's success depends on the actions of others. It can be applied to actual games, economics, military strategy, business strategy, relationships, and more.

Let's do an example in football. As the coach of the offensive team, you have the choice to pass or run. The defensive team has the option to defend versus a pass, or defend versus a run, depending on what they think the offense will do. As offensive coach, you want to model the strategies so that you can maximize our return in terms of yards gained per play. Consider the following payoff matrix, which shows how many average yards the offense will gain in each scenario:



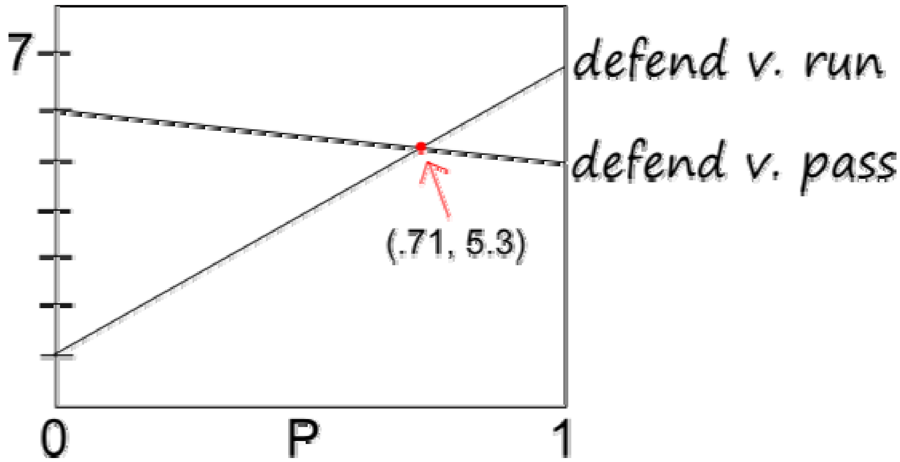
There is no one strategy that will always work; otherwise the game would not be very interesting. If you always pass or always run, then the defense will always be able to defend against you. Rather, the best strategy is to choose each option with a certain probability. Let's assign the variable P to the probability that you (the offense) will pass. Then, if the defense is set to defend against a pass, the expected value for the number of yards gained is

$$EV\_vp = (5)(P) + (6)(1-P) = 6-P$$

If the defense is planning to defend against a run, then

$$EV\_vr = (7)(P) + (1)(1-P) = 6P +1$$

Now, let's graph the expected gain as a function of the probability of passing:



We can find the intersection of these graphs by setting the expected values equal to one another:

EV_vp = EV_vr
6-P = 6P +1
5 = 7P
P=5/7 = 71.4%

So the optimal strategy is to pass 71% of the time, and run 29% of the time, which will give an optimal expected minimum gain of 5.3 yards. If you do anything else, the defense can act in such a way to lower your average gain.

We can repeat this calculation from the defense's point of view, and we get that the optimal strategy for them is to defend versus a pass 86% of the time, and defend versus a run 14% of the time.

This solution is known as a **NASH EQUILIBRUIM** strategy, so named after the Nobel Prize-winning economist John Nash (played by Russell Crowe in the film "A Beautiful Mind").

Simply, a Nash Equilibrium is a solution to the game such that

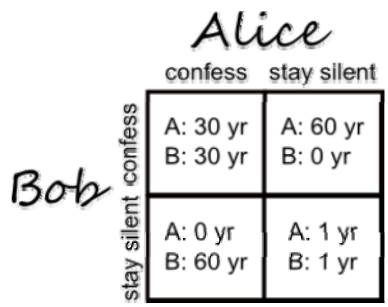
- each player knows the equilibrium strategy of the other, and
- neither player will benefit by changing his strategy.

In our football example, the defense is making the best decision it can, based on the offense's actions, and the offense is making the best decision it can based on the defense's actions. (Yes, we realize that real football is actually much more complicated, as the defense can often guess your strategy by who is on the field, the distribution of the players, watching hours of video tapes, etc. However, the concept still applies - a good coach or quarterback intuits these probabilities and acts accordingly. But we digress.)

One important thing to note about the Nash Equilibrium is that it does not necessarily represent the optimal outcome for everyone. Let's go to an example where this is the case.

The Prisoner's Dilemma

The Prisoner's Dilemma is the classic game theory problem, in which two people, call them Alice and Bob, are arrested for a pair of crimes: murder and illegal weapons possession. The police have enough evidence to convict them both for the possession crime, with a light sentence of 1 year. They do not, however, have enough evidence to convict either on the murder charge, without a confession from the other. They are offering to release whoever rats out the other. That is, if both remain silent, then they will both get just one year in jail. However, each risks getting life in prison (let's say 60 years) if the other also takes the deal. We can represent this dilemma with the following diagram.



That is, the best scenario for both people is if both stay silent and just take their 1 year for possession. If both of them talk, they both get a 30-year sentence. If just Alice talks, then she gets to go free and Bob gets the 60-year sentence, and vice versa.

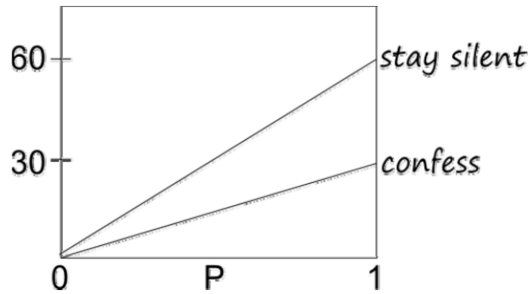
Let's analyze mathematically what Alice should do. Say Bob will confess with probability P. Then, if Alice confesses, the expected value of her sentence is given by:

$$\text{Expected Value Confess} = (P)(30) + (1-P)(0) = 30P$$

If she stays silent, the expected value of her sentence is:

$$\text{Expected Value Silent} = (P)(60) + (1-P)(1) = 59P + 1$$

We can graph both of these equations, where the x-axis represents Bob's probability of confessing (P) and the y-axis represents the expected value of Alice's sentence.



These two lines cross where Expected Value Confess equals Expected Value Silent.

$30P = 59P+1$

$-29 P = 1$

$P= -1/29$

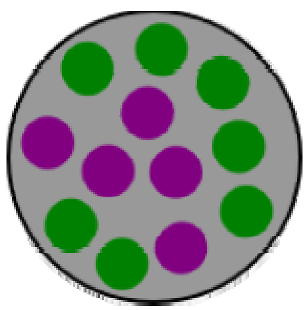
A negative probability is out of our range. This means that she should ALWAYS confess. This is the Nash Equilibrium for this problem. Note that both Alice and Bob would be better off if they both stayed silent, but neither knows that the other won't confess.

The Prisoner's Dilemma has many practical applications. Say I'm opening a cupcake store across the street from a rival cupcake store. I have two options: cut prices or do not cut prices. If the other store does not cut prices, then I should cut my prices in order to gain customers. However, if the other store does cut prices, then I should cut my prices in order to keep my customers. So I should always cut prices. However, both my competitor and I would be better off if neither of us cut prices. It can also be applied in military situations: if there is a chance of war, should we strike first? Everyone is better off if neither strikes, but…

Conditional Probability

So far, we have only considered situations in which each experiment is independent of the results of other experiments. But I can also ask the question "What is the probability of event A happening, given that event B has already happened?" This leads us to the field of Conditional Probability.

As a simple example, assume we have a jar with 5 purple and 7 green balls, like so:



Jar with 7 green balls and 5 purple balls.

We will reach in and choose 2 balls at random, one after another, without replacing the first. You ask me: What is the probability that the second ball will be purple? All I can really say right now is: Well, it depends. It depends on what the first ball was.

The probability that the first ball is purple is 5/12, since there are 5 purple balls and 12 total. After we choose the first ball, there are 11 balls left. If the first ball was purple, we know there are now 4 purple and 7 green left. Then the probability of picking a second purple is 4/11. The total probability is the product of these:

$$P(\text{purple, purple}) = (5/12)(4/11) = 20/132 \text{ or about } 15\%.$$

If the first ball was green (probability 7/12), then there are 5 purple and 11 total balls left. So the probability the second one being purple is:

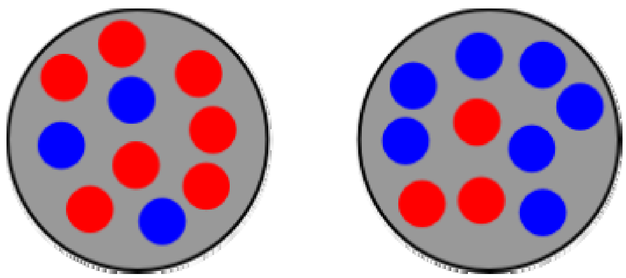
$$P(\text{green,purple}) = (7/12)(5/11) = 35/132 \text{ or about } 26\%.$$

Definition #10: CONDITIONAL PROBABILITY - the probability of event A happening, given that event B has already happened.

The Two Jars Problem

Say we have two jars, each with 10 balls. One jar has 7 red balls and 3 blue, while the other has 7 blue balls and 3 red. You can pick one ball at a time, but have to throw it back in when you are done. If you guess which is the Red-Majority-Jar, you win $1000. But each time you take out one ball, the prize drops $50.

We can use probabilities to maximize your chances of winning the biggest prize. First, don't pick any balls. Chances that you randomly pick the right jar is 50%, or 1 to 1, and the prize is $1,000.



Two Jars - The Red-Majority Jar and the Blue Majority Jar

Ok, now you pick one ball from one jar. The prize drops to $950. Say the ball is red. How strong is the evidence that this is the Red-Majority-Jar?

The odds of picking a red from the Red-Majority Jar is 7/10 or 70%
The odds of picking a red from the Blue-Majority Jar is 3/10 or 30%
Odds of winning are now 7 to 3, or 70%.

Not good enough? OK, put the ball back in, shake up the jar, and pick again. Say you get red again. The prize is down to $900. Now how strong is your evidence?

Chances of picking two reds in a row from Red-Majority-Jar = (.7)(.7) = 49%
Chances of picking two reds in a row from Blue-Majority-Jar = (.3)(.3) = 9%
Odds are now 5 to 1 or about 84%.

But what happens if the second ball was blue?

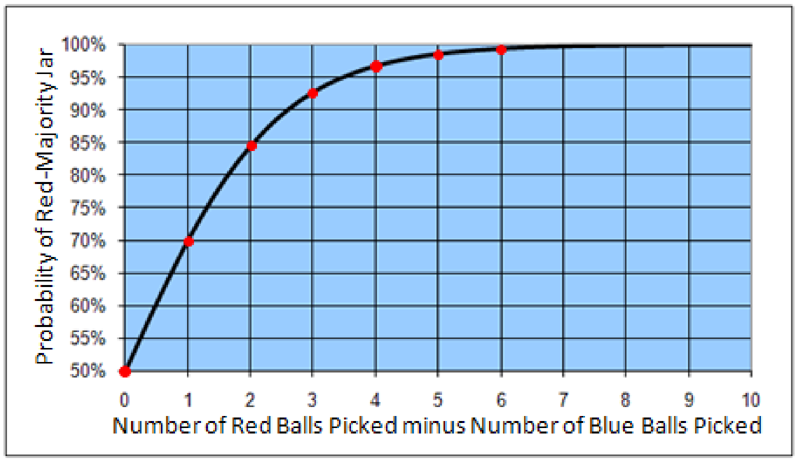
Chances of picking a red and a blue from Red-Majority-Jar = (.7)(.3) = 21%
Chances of picking a red and a blue from Blue-Majority-Jar = (.3)(.7) =21%
So the odds are now back to 1 to 1 or 50%.

Once more? OK, the prize is down to $850. Let's assume the first two were red, and now you picked another red.

Chances of picking three reds in a row from Red-Majority-Jar = (.7)(.7)(.7) = 34%
Chances of picking three reds in a row from Blue-Majority-Jar = (.3)(.3(.3)) = 2.7%
Your odds are now near 12.5 to 1, or about 93%

But what happens if that third ball is blue? You do the math. Hint: the answer is 70%, same as if you'd only picked one ball.

It turns out that it actually doesn't matter how many times you pick. The probability is based solely on how many MORE red balls you picked than blue balls. The plot looks like this:



The Mathematics of the Two Jar Problem

For example, if you picked 50 times and got 27 reds and 23 blues. What is the probability that you were picking from the red jar? The formula looks like this:

P (red jar | 27r 23 b) = (0.7)^27 (0.3)^23 / [(0.7)^27 (0.3)^23 + (0.3)^27 (0.7)^23] = 97%

Or you can use the chart: 27-23 = 4 more reds than blues = 97%. That's the same as if you'd picked 4 reds and no blues, or 6 reds and 2 blues, or 104 reds and 100 blues.

The point here is that history does matter. Your chance of correctly guessing if you've got the red-majority-jar is totally dependent on your history of picking balls.

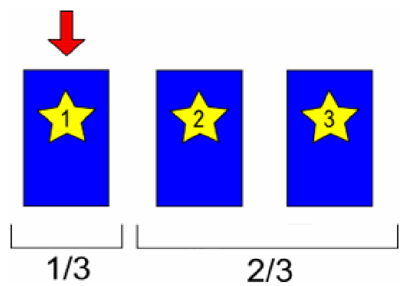
Conditional Probability - The Monty Hall Problem

There was a popular game show in the 1970's, called Let's Make a Deal, hosted by a man named Monty Hall, who was not related to Monty Python. The game is played as follows: there are three doors. Behind two of the doors are goats. Behind the third door is a brand new car. Of course you don't know which door has the car, and we assume you'd rather have the car than a goat. First you get to pick a door, but not open it. Then, Monty opens one of the remaining two doors and reveals a goat.

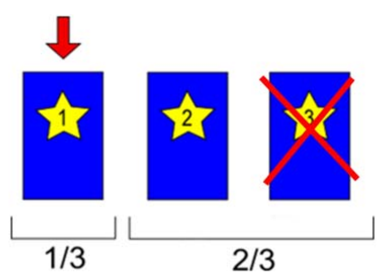
He then gives you a choice: stay with your original door, or change to the other closed door. The question is, should you change or stay? Which gives you the higher probability of winning? We assume you can ignore the loud yelling from the audience and the ticking clock counting down the 7 seconds you have to decide.

But of course you have already thought this through long before you got on the show, right? Since each door has a 1 in 3 chance of being the winning door, it doesn't matter if you stay or change. This is true that at the start of the game. Each door is equally likely. But, after Monty opens a door to expose a goat, the game is totally changed. In fact, you are twice as likely to win the car if you change doors as if you stay!

Here is the logic. Say you pick door #1. There is a 1/3 chance that door #1 is the winning door, and a 2/3 chance that the winning door is not the one you chose. That is, a 2/3 chance that the winner is door #2 or door #3. Originally, chance was evenly distributed over the doors, like so:



However, Now Monty opens door #3, which he knows is not the car, so that 2/3 chance is no longer evenly distributed. The entire 2/3 chance is now at door #2, like so:



So door #1 still has a 1/3 chance of winning, but door #2 has a 2/3 chance of winning. So by switching, you have doubled your chances of winning.

Take this to the extreme if you are still confused. What if there are one million doors, and after you pick one, Monty Hall opens all but one of the rest of them. Remember he knows which one has the car, so he won't open that one! The chance of your original choice being the wining door is still 1/1,000,000, but the chance of the other door being the car is now 999,999/1,000,000. I would switch.

Of course in the actual show, Monty Hall did not always offer you the option to switch. Sometimes, he would just let you pick a door, and then show you your goat (or sometimes your car!). Other times, he would not only give you the option of switching, but even offer you money to switch. Of course, this made many contestants suspicious that they had chosen the correct door, and he was just trying to lure them away. In an interview with the New York Times in 1991, Monty Hall said "If the host is required to open a door and offers you a switch, then you should take the switch, but if he has the choice whether to allow a switch or not, beware. Caveat emptor. It all depends on his mood. My only advice is, if you can get me to offer you $5,000 to not open the door, take the money and go home."

Reference: http://www.nytimes.com/1991/07/21/us/behind-monty-hall-s-doors-puzzle-debate-and-answer.html?pagewanted=5&src=pm

Bayesian Probability

Frequentist Probability describes the methods we have been applying thus far, in which we can determine probability based on a large number of trials.

We now look at a different model, called Bayesian Probability, based on the work of an 18th century mathematician and theologian, Thomas Bayes (1702–1761), and made popular by the French mathematician Pierre-Simon Laplace (1749–1827).

In the Bayesian model, probabilities are viewed as more of a measure of how much we know, rather than a chance of occurrence. We all apply this method in our everyday lives. For example, if your friend says that there is a 70% chance that she will make it to your birthday party; you probably wouldn't envision 100 clones of your friend and 100 birthday parties, of which she shows up to 70. The "70% chance" is more of a statement about the strength of her belief, rather than what would happen in a large number of trials. If her car breaks down the day before your party, then she will probably adjust that prediction accordingly.

Why Your Doctor is That Way

In the Bayesian Procedure, we start with some information about a situation and a general probability model. We then update the model as we collect new data. The general steps are:

- List possible hypotheses with initial estimates of their probabilities.
- Gather information about the situation, do tests or experiments if possible.
- Update the probability estimates using the new information.

Let's see how we can apply this procedure. We have a collection of possible states, and a sense of what the relative probability of each is, so we assign to each state a value, such that the sum of all the possible values is 1 (100%). For example, you go to the doctor with some symptoms, and he says that based on the information that he has, there is a 50% chance that you have disease A, a 40% chance of disease B, and 10% chance of disease C. We would notate this as:

$$P(A) = 50\%$$
$$P(B) = 40\%$$
$$P(C) = 10\%$$

He orders a blood test. Your white cell count turns out to be is low. Let’s call that symptom L.

The medical literature says that 10% of people with disease A have symptom L, 60% of people with B have it, and 30% of C have it. He can now update his predictions accordingly. Let's use the notation for conditional probability:

$$P(L\,|A\,) = 10\%$$
$$P(L\,|\,B) = 60\%$$
$$P(L\,|\,C) = 30\%$$

So this means that if there is a 50% chance you have disease A, and a 10% chance of symptom L given that you have disease A, so the probability of having both is the product (50%)(10%) = 5%. We can repeat this for B and C to get the fourth column of the table below.

Disease (X)	P(X)	P(Q | X)	P(X) P(Q|X)	Updated P(X)
A	50%	10%	5%	5/32 = **16%**
B	40%	60%	24%	24/32 = **75%**
C	10%	30%	3%	3/32 = **9%**
			SUM = 32%	

The doctor updates his prediction by dividing each revised probability in column four by the total (32%) to get the numbers in the last column. In this case, by having the (cheap) blood test, the doctor now knows that there is a 75% chance you have disease B, versus 16% for A and 9% for C. Of course, this is still a probability distribution, not truth: you may well still have either disease A, B or C, but at this point your doctor may opt to treat disease B and see if you get better.

Note that there may well be an expensive test out there to determine without a doubt whether you have disease A, B or C, but the doctor has to weigh the cost of that test versus managed care issues, insurance reimbursements, etc. The fact is that medicine has become more and more probabilistically-based than most of us realize. But that’s another discussion.

Summary and Conclusion

Probability Theory has its roots in determining odds for games of chance. But as time goes on, Probability Theory has found its way into many fields. It already plays a fundamental role in our understanding of the physical world around us, from biology to physics to our health and our stock portfolios. Randomness and uncertainty are everywhere, and probability gives us a quantitative way to deal with them in such a way that we can make informed decisions. At best, probability often confronts us with counterintuitive ideas that make us re-evaluate what we think we know. And at worst, just because we make a ‘good’ decision based on a probabilistic analysis, doesn't mean we won’t be disappointed at the actual outcome.

One major application of probability that we’ve skipped here is in the field of statistics. One can make a probabilistic model of what one might expect from an experiment, and then compare that to a given a set of data. If there is a large difference between them, you go back and change your model accordingly. The use of probability in statistics is covered in another of our courses: **Understanding Statistics -, What it is, Its Proper Use, and Its Widespread Misuse**.