PDHonline Course E139 (6 PDH)

Automatic Control Systems - Part II: Laplace Transform and Time-Domain Analysis

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Automatic Control Systems

Part II:

Laplace Transform and Time-Domain Analysis

By Shih-Min Hsu, Ph.D., P.E.
IV. Laplace Transform:

The Laplace transform is one of the mathematical tools used for the solution of ordinary linear differential equations. The Laplace transform method has the following two attractive features:

1. The homogeneous equation and the particular integral are solved in one operation.
2. The Laplace transform converts the differential equation into an algebraic equation in $s$. It is possible to manipulate the algebraic equation by simple algebraic rules to obtain the solution in the $s$-domain. The final solution is obtained by taking the inverse Laplace transform.

Definition of the Laplace Transform:

Given the function $f(t)$ that satisfies the condition:

$$\int_0^\infty f(t)e^{-\sigma t} \, dt < \infty,$$  \hspace{1cm} (4-1)

for some finite real $\sigma$, the Laplace transform of $f(t)$ is defined as

$$F(s) = \int_0^\infty f(t)e^{-st} \, dt,$$  \hspace{1cm} (4-2)

or

$$F(s) = \mathcal{L}[f(t)].$$  \hspace{1cm} (4-3)

The variable $s$ is referred to as the Laplace operator, which is a complex variable, $s = \sigma + j\omega$, where $\sigma$ is the real part and $\omega$ is the imaginary part, as shown in Figure 4-1.

Fig. 4-1. Graphical presentation of Laplace operator ($s$-Plane).
Example 4-1: Let \( f(t) \) be a unit step function that is defined to have a constant value of unity for \( t \geq 0 \) and a zero for \( t < 0 \), namely, \( f(t) = u(t) \). What is its Laplace transform \( F(s) \)?

Solution:

\[
F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty u(t)e^{-st}dt = -\frac{1}{s}e^{-st}\bigg|_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}
\]

Example 4-2: Let \( f(t) \) is an exponential function, \( f(t) = e^{-at} \) \( t \geq 0 \), where \( a \) is a constant. What is its Laplace transform \( F(s) \)?

Solution:

\[
F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = -\frac{1}{s+a}e^{-(s+a)t}\bigg|_0^\infty = \frac{1}{s+a}
\]
Important Theorems of the Laplace Transform:

1. **Multiplication by a constant**-
   \[ L[kf(t)] = kF(s) \]  \hspace{1cm} (4-4)

2. **Sum and difference**-
   \[ L[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s) \]  \hspace{1cm} (4-5)

3. **Differentiation**-
   \[
   \begin{align*}
   L\left[ \frac{df(t)}{dt} \right] &= sF(s) - \lim_{t \to 0} f(t) = sF(s) - f(0) \\
   L\left[ \frac{d^n f(t)}{dt^n} \right] &= s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)
   \end{align*}
   \]  \hspace{1cm} (4-6) (4-7)

   where \( f^{(k)}(0) = \frac{d^k f(t)}{dt^k} \bigg|_{t=0} \).

4. **Integration**-
   \[
   \begin{align*}
   L\left[ \int_0^t f(\tau)d\tau \right] &= \frac{F(s)}{s} \\
   L\left[ \int_0^t \int_0^{\tau_2} \cdots \int_0^{\tau_{n-1}} f(\tau)d\tau \ d\tau_1 \ d\tau_2 \cdots d\tau_{n-2} \ d\tau_{n-1} \right] &= \frac{F(s)}{s^n}
   \end{align*}
   \]  \hspace{1cm} (4-8) (4-9)

5. **Shift in time**-
   \[ L[f(t-T)u(t-T)] = e^{-sT}F(s) \]  \hspace{1cm} (4-10)

6. **Initial-value theorem**-
   \[ \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s) \]  \hspace{1cm} (4-11)

7. **Final-value theorem**-
   \[ \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \]  \hspace{1cm} (4-12)

8. **Complex shifting**-
   \[ L[e^{at}f(t)] = F(s - a) \]  \hspace{1cm} (4-13)

9. **Real convolution (Complex multiplication)**-
   \[ F_1(s)F_2(s) = L\left[ \int_0^t f_1(\tau)f_2(t-\tau)d\tau \right] = L\left[ \int_0^t f_2(\tau)f_1(t-\tau)d\tau \right] = L[f_1(t)*f_2(t)] \]  \hspace{1cm} (4-14)

   where \( * \) denotes complex convolution.
Inverse Laplace Transformation:

The operation of obtaining \(f(t)\) from the Laplace transform \(F(s)\) is called the inverse Laplace transformation. The inverse Laplace transform of \(F(s)\) is expressed as follows

\[
f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds ,
\]

where \(c\) is a real constant that is greater than the real parts of all the singularities of \(F(s)\). Equation (4-15) represents a line integral that is to be evaluated in the \(s\)-plane. However, for most engineering purposes the inverse Laplace transform operation can be done simply by referring to the Laplace transform table, such as the one given in Table 4-1. Before using Table 4-1, one may need to do partial-fraction expansion first, then, use the equations in table 4-1.

Partial-fraction expansion when all the poles\(^(*)\) of the transfer function are simple and real:

Example 4-3: Find the partial-fraction expansion of the following transfer function, and its inverse Laplace.

\[G(s) = \frac{s + 3}{(s + 1)(s + 2)}\]

Solution:

\[G(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2}\]

where,

\[A = (s + 1)G(s) \bigg|_{s = -1} = \frac{s + 3}{s + 2} \bigg|_{s = -1} = \frac{-1 + 3}{-1 + 2} = 2\]

and

\[B = (s + 2)G(s) \bigg|_{s = -2} = \frac{s + 3}{s + 1} \bigg|_{s = -2} = \frac{-2 + 3}{-2 + 1} = -1\]

Therefore,

\[G(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{2}{s + 1} + \frac{-1}{s + 2}\]

The inverse Laplace transform of the given transfer function can be obtained by using some equation in Table 4-1, namely,

\[g(t) = \mathcal{L}^{-1}\left(\frac{2}{s + 1} + \frac{-1}{s + 2}\right) = \mathcal{L}^{-1}\left(\frac{2}{s + 1}\right) + \mathcal{L}^{-1}\left(\frac{-1}{s + 2}\right) = 2e^{-t} - 1e^{-2t} \quad t \geq 0\]

\(\star\)

Note (*): A pole is a value of \(s\) that makes a function, such a \(G(s)\), infinite, by making the denominator of the function to zero.
Partial-fraction expansion when some poles of the transfer function are of multiple order:

Example 4-4: Find the partial-fraction expansion of the following transfer function, and its inverse Laplace.

\[ G(s) = \frac{1}{s^2(s+1)} \]

Solution:

\[ G(s) = \frac{1}{s^2(s+1)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{B}{s+1} \]

where,

\[ A_2 = [s^2G(s)]_{s=0} = \left[ \frac{1}{s+1} \right]_{s=0} = 1 \]

\[ A_1 = \frac{d}{ds} [s^2G(s)]_{s=0} = \frac{d}{ds} \left[ \frac{1}{s+1} \right]_{s=0} = \left[ \frac{-1}{(s+1)^2} \right]_{s=0} = -1 \]

and

\[ B = (s+1)G(s) \bigg|_{s=-1} = \frac{1}{s^2} \bigg|_{s=-1} = 1 \]

Therefore,

\[ G(s) = \frac{1}{s^2(s+1)} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \]

The inverse Laplace of the given transfer function can be obtained by using equations in Table 4-1, namely,

\[ g(t) = \mathcal{L}^{-1} \left( -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \right) = \mathcal{L}^{-1} \left( -\frac{1}{s} \right) + \mathcal{L}^{-1} \left( \frac{1}{s^2} \right) + \mathcal{L}^{-1} \left( \frac{1}{s+1} \right) = -1 + t + e^{-t} \quad t \geq 0 \]

Example 4-5: Find the partial-fraction expansion of the following transfer function, and its inverse Laplace.

\[ G(s) = \frac{1}{s(s+1)^3(s+2)} \]

Solution:

\[ G(s) = \frac{1}{s(s+1)^3(s+2)} = \frac{A_1}{s} + \frac{A_2}{(s+1)} + \frac{A_3}{(s+1)^2} + \frac{B}{s+2} + \frac{C}{s} \]

where
Partial-fraction expansion with simple complex-conjugate poles:

Example 4-6: Find the partial-fraction expansion of the following transfer function, and its inverse Laplace.

\[
G(s) = \frac{\omega_n^3}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}
\]

Solution:
\[ G(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{s(s + \alpha - j\omega)(s + \alpha + j\omega)} = \frac{A_1}{s + \alpha - j\omega} + \frac{A_2}{s + \alpha + j\omega} + \frac{B}{s} \]

and

\[ \alpha = \zeta \omega_n \quad \omega = \omega_n \sqrt{1 - \zeta^2} \]

where

\[ A_1 = (s + \alpha - j\omega)G(s) \big|_{s=-\alpha+j\omega} = \frac{\omega_n^2}{s(s + \alpha + j\omega)} \big|_{s=-\alpha+j\omega} = \frac{\omega_n^2}{2\omega} e^{-\left(\frac{\theta + \pi}{2}\right)} \]

\[ A_2 = (s + \alpha + j\omega)G(s) \big|_{s=-\alpha-j\omega} = \frac{\omega_n^2}{s(s + \alpha - j\omega)} \big|_{s=-\alpha-j\omega} = \frac{\omega_n^2}{2\omega} e^{-\left(\frac{\theta + \pi}{2}\right)} \]

with \( \theta = \tan^{-1}\left[\frac{\omega}{-\alpha}\right] \) (please note this angle is in the second quadrant)

and

\[ B = sG(s) \big|_{s=0} = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \big|_{s=0} = 1 \]

Therefore,

\[ G(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{2\omega} \left[ e^{-\left(\frac{\theta + \pi}{2}\right)} + e^{-\left(\frac{\theta - \pi}{2}\right)} \right] \frac{1}{s} \]

The inverse Laplace of the given transfer function can be obtained by using Table 4-1, namely,

\[ g(t) = 1 + \frac{\omega_n}{2\omega} \left[ e^{-\left(\frac{\pi}{2}\right)} e^{-\left(-\alpha+j\omega\right)t} + e^{-\left(\frac{\pi}{2}\right)} e^{-\left(-\alpha-j\omega\right)t} \right] = 1 + \frac{\omega_n}{2\omega} e^{-\alpha t} \left[ e^{-\left(\frac{\pi}{2}\right) - \omega t} + e^{-\left(\frac{\pi}{2}\right) - \omega t} \right] \]

\[ = 1 + \frac{\omega_n}{\omega} e^{-\omega t} \left[ \frac{e^{-\left(\omega t - \frac{\pi}{2}\right)}}{2} + e^{-j\left(\omega t - \frac{\pi}{2}\right)} \right] = 1 + \frac{\omega_n}{\omega} e^{-\omega t} \left[ e^{-j(\omega t - \theta)} - e^{-j(\omega t - \theta)} \right] = 1 + \frac{\omega_n}{\omega} e^{-\omega t} \sin(\omega t - \theta) \]

\[ = 1 + \frac{1}{\sqrt{1 - \zeta^2}} e^{-\omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2} t - \theta \right) \quad t \geq 0 \]

where

\[ \theta = \tan^{-1}\left[\frac{\sqrt{1 - \zeta^2}}{-\zeta}\right] \]
Laplace Transform Table:

Table 4-1: Laplace Transform Table.

<table>
<thead>
<tr>
<th>Laplace Transform, ( F(s) )</th>
<th>Time Function, ( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \delta(t) )</td>
</tr>
<tr>
<td>( \frac{1}{s} )</td>
<td>( u_s(t) )</td>
</tr>
<tr>
<td>( \frac{1}{s^2} )</td>
<td>( t )</td>
</tr>
<tr>
<td>( \frac{n!}{s^{n+1}} )</td>
<td>( t^n )</td>
</tr>
<tr>
<td>( \frac{1}{s + a} )</td>
<td>( e^{-at} )</td>
</tr>
<tr>
<td>( \frac{1}{(s + a)(s + b)} )</td>
<td>( \frac{e^{-at} - e^{-bt}}{b - a} )</td>
</tr>
<tr>
<td>( \frac{\omega_n}{s^2 + \omega_n^2} )</td>
<td>( \sin \omega_n t )</td>
</tr>
<tr>
<td>( \frac{s}{s^2 + \omega_n^2} )</td>
<td>( \cos \omega_n t )</td>
</tr>
<tr>
<td>( \frac{1}{(s + a)^2} )</td>
<td>( te^{-at} )</td>
</tr>
<tr>
<td>( \frac{n!}{(s + a)^{n+1}} )</td>
<td>( t^n e^{-at} )</td>
</tr>
<tr>
<td>( \frac{\omega_n}{(s + a)^2 + \omega_n^2} )</td>
<td>( e^{-at} \sin \omega_n t )</td>
</tr>
<tr>
<td>( \frac{s + a}{(s + a)^2 + \omega_n^2} )</td>
<td>( e^{-at} \cos \omega_n t )</td>
</tr>
</tbody>
</table>
\[
\begin{array}{|c|c|}
\hline
\frac{1}{(1 + sT)^n} & \frac{1}{T^n(n-1)!} e^{-\frac{t}{T}} \\
\hline
\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} & \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\phi} \sin \omega_n \sqrt{1 - \zeta^2} t \\
\hline
\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} & 1 + \frac{1}{\sqrt{1 - \zeta^2}} e^{-\phi} \sin \left(\omega_n \sqrt{1 - \zeta^2} t - \phi\right) \\
& \text{where } \phi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{-\zeta} \\
\hline
\frac{s \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} & \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\phi} \sin \left(\omega_n \sqrt{1 - \zeta^2} t + \phi\right) \\
& \text{where } \phi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{-\zeta} \\
\hline
\frac{1}{s(1 + sT)} & 1 - e^{-\frac{t}{T}} \\
\hline
\frac{1}{s(1 + sT)^2} & 1 - \frac{t + T}{T} e^{-\frac{t}{T}} \\
\hline
\frac{1}{s^2(1 + sT)^2} & t - 2T + (t + 2T)e^{-\frac{t}{T}} \\
\hline
\frac{1 + as}{s^2(1 + sT)} & t + (a - T) \left(1 - e^{-\frac{t}{T}}\right) \\
\hline
\frac{s}{(s^2 + \omega_n^2)^2} & \frac{1}{2\omega_n} t \sin \omega_n t \\
\hline
\frac{s}{(s^2 + \omega_{n1}^2)(s^2 + \omega_{n2}^2)} & \frac{1}{\omega_{n2}^2 - \omega_{n1}^2} \left(\cos \omega_{n1} t - \cos \omega_{n2} t\right) \\
\hline
\frac{s^2}{(s^2 + \omega_n^2)^2} & \frac{1}{2\omega_n} \left(\sin \omega_n t + \omega_n t \cos \omega_n t\right) \\
\hline
\end{array}
\]
V. Time-Domain Analysis:

The time response of a control system is usually divided into two parts: the transient response and the steady-state response. Let \( c(t) \) denote a time response and be expressed as follows:

\[
c(t) = c_t(t) + c_{ss}(t),
\]

where \( c_t(t) \) is the transient response and \( c_{ss}(t) \) is the steady-state response.

Typical Test Signals:

To facilitate the time-domain analysis, the following input test signals are often used.

**Step Function:** The step function represents an instantaneous change in the reference input variable. At time \( t < 0 \), the signal is zero while at \( t \geq 0 \), the signal is \( R \), namely,

\[
r(t) = \begin{cases} 
R & t \geq 0 \\
0 & t < 0 
\end{cases},
\]

where \( R \) is a constant. Or,

\[
r(t) = Ru_s(t),
\]

where \( u_s(t) \) is the unit step function. The step function as a function of time is shown in Figure 5-1.

![Fig. 5-1. Step function input.](image)

**Ramp Function:** The ramp function is a signal to have a constant change in value at the same rate with respect to time and can be expressed as

\[
r(t) = \begin{cases} 
Rt & t \geq 0 \\
0 & t < 0 
\end{cases}.
\]
As one can see, the slope of the signal is a constant, $R$. The ramp function is shown in Figure 5-2.

\[ r(t) = \begin{cases} \frac{R}{2} t^2 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (5-5) \]

The graphical representation of the parabolic function is shown in Figure 5-3.

**Parabolic Function:** The mathematical representation of a parabolic function is expressed as follows:

The graphical representation of the parabolic function is shown in Figure 5-3.

**Steady-State Error of Linear Systems:**

A closed-loop system with a negative feedback is shown in Figure 5-4. The output can be expressed as a function of the input as follows:
\[ C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s). \]  
\[ E(s) = R(s) - H(s)C(s), \]  
\[ E(s) = \left[ 1 - \frac{G(s)H(s)}{1 + G(s)H(s)} \right] R(s) = \frac{1}{1 + G(s)H(s)} \cdot R(s). \]

The steady-state error in time domain can be obtained by applying the final-value theorem, namely,

\[ e_{ss}(t) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \cdot \frac{1}{1 + G(s)H(s)} \cdot R(s). \]

Typically, the inputs for steady-state error considerations are:

1. If the input is a step function, \( u(t) \), of magnitude \( R \), or \( \frac{R}{s} \) in Laplace transform. Then, the steady-state error of the system can be denoted as

\[ e_{ss}(t) = \lim_{s \to 0} s \cdot \frac{1}{1 + G(s)H(s)} \cdot R(s) = \lim_{s \to 0} s \cdot \frac{1}{1 + G(s)H(s)} \cdot \frac{R}{s} = \lim_{s \to 0} \frac{R}{1 + G(s)H(s)}. \]  

If one defines

\[ K_p = \lim_{s \to 0} G(s)H(s), \]

where \( K_p \) is the step error constant (or position error constant). Then, the equation (5-10) can be simplified as

\[ e_{ss}(t) = \lim_{s \to 0} \frac{R}{1 + G(s)H(s)} = \frac{R}{1 + \lim_{s \to 0} G(s)H(s)} = \frac{R}{1 + K_p}. \]

One can easily realize that to have the steady-state error to be zero when the input is a step function, \( K_p \) must be infinite. The steady-state error due to a unit step function input is also called the steady-state position error.
(2) If the input is a ramp function, its Laplace transform can be expressed as
\[ R(s) = \frac{R}{s^2}. \] (5-13)

Then, the steady-state error of the system is
\[ e_{ss}(t) = \lim_{s \to 0} \frac{1}{1 + G(s)H(s)} \cdot \frac{R}{s^2} = \lim_{s \to 0} \frac{1}{s + sG(s)H(s)} \cdot \frac{R}{\lim_{s \to 0} G(s)H(s)}. \] (5-14)

If one defines
\[ K_v = \lim_{s \to 0} sG(s)H(s), \] (5-15)
where \( K_v \) is the ramp error constant (or velocity error constant). Therefore, the equation (5-14) can be simplified as
\[ e_{ss}(t) = \frac{R}{K_v}. \] (5-16)

Therefore, to have zero steady-state error, \( K_v \) needs to be infinite. The steady-state error due to a unit ramp function input is called the steady-state velocity error.

(3) If the input is a parabolic input, its transfer function is listed below
\[ R(s) = \frac{R}{s^3}. \] (5-17)

The steady-state error of the system can be obtained as follows
\[ e_{ss}(t) = \lim_{s \to 0} \frac{1}{1 + G(s)H(s)} \cdot \frac{R}{s^3} = \lim_{s \to 0} \frac{1}{s + s^2G(s)H(s)} \cdot \frac{R}{\lim_{s \to 0} s^2G(s)H(s)}. \] (5-18)

If one defines the parabolic error constant (or acceleration error constant),
\[ K_a = \lim_{s \to 0} s^2G(s)H(s). \] (5-19)

Then, the steady-state error due to a parabolic input can be written as
\[ e_{ss}(t) = \frac{R}{K_a}, \] (5-20)
and it is also called the steady-state acceleration error.

Example 5-1: A closed-loop system is shown below.

[Diagram of a closed-loop system with the transfer function \( \frac{K(s + 6)}{s(s + 2)(s + 4)} \) and input \( R(s) \) and output \( C(s) \).]
(A) For \( K = 10 \), what is the steady-state error to a unit step input?
(B) For \( K = 10 \), what is the steady-state error to a unit ramp input?
(C) For \( K = 10 \), what is the steady-state error to a unit parabolic input?
(D) Find the value of \( K \) such that the steady-state error to a unit ramp input is 5%.

Solution:

(A) For a unit step input and \( K = 10 \):

\[
K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \left[ \frac{10(s + 6)}{s(s + 2)(s + 4)} \right] \cdot 1 = \infty.
\]

\[
e_{ss}(t) = \frac{1}{1 + K_p} = \frac{1}{1 + \infty} = 0.
\]

(B) For a unit ramp input and \( K = 10 \):

\[
K_v = \lim_{s \to 0} s G(s)H(s) = \lim_{s \to 0} \left[ \frac{10(s + 6)}{s(s + 2)(s + 4)} \right] \cdot 1 = \lim_{s \to 0} \frac{10(s + 6)}{(s + 2)(s + 4)} = \frac{60}{8} = 7.5
\]

\[
e_{ss}(t) = \frac{1}{K_v} = \frac{1}{7.5} = 0.1333 = 13.33\%.
\]

(C) For a Parabolic input and \( K = 10 \):

\[
K_a = \lim_{s \to 0} s^2 G(s)H(s) = \lim_{s \to 0} \left[ \frac{10(s + 6)}{s(s + 2)(s + 4)} \right] \cdot 1 = \lim_{s \to 0} \frac{10(s + 6)}{(s + 2)(s + 4)} = 0.
\]

\[
e_{ss}(t) = \frac{1}{K_a} = \frac{1}{0} = \infty.
\]

(D) From part (B), for a unit ramp input,

\[
K_v = \lim_{s \to 0} s G(s)H(s) = \lim_{s \to 0} \left[ \frac{K(s + 6)}{s(s + 2)(s + 4)} \right] \cdot 1 = \lim_{s \to 0} \frac{K(s + 6)}{(s + 2)(s + 4)} = \frac{6K}{8}
\]

\[
e_{ss}(t) = \frac{1}{K_v} = \frac{1}{6K/8} = 5\% = 0.05.
\]

\[
K = \frac{8}{6 \cdot (0.05)} = \frac{80}{3}.
\]

Type of Feedback Control System:

A stable system can be classified according to the degree of the polynomial for which the error is a constant, and the classification is called the system type. In general, the open-loop transfer function \( G(s)H(s) \) can be written as

\[
G(s)H(s) = \frac{K(1 + a_1 s)(1 + a_2 s) \cdots (1 + a_m s)}{s^T (1 + b_1 s)(1 + b_2 s) \cdots (1 + b_n s)},
\]

where that \( K \), all \( a \)’s and \( b \)’s are constants and the \( T = 0, 1, 2, \ldots \). The type of feedback control system is referring to the order of the pole of \( G(s)H(s) \) at \( s = 0 \). Therefore, the system is of type \( T \). For instance, a feedback control system with
\[ G(s)H(s) = \frac{5(1 + 2s)}{s(1 + 3s)(1 + 4s)} \]
is of type 1.

The significance of system types is as follows (when restricted to unity feedback systems):

- **Type 0**: System has a non-zero steady-state error to a step input.
- **Type 1**: System has zero steady-state error to a step input. It has a non-zero steady-state error to a ramp input.
- **Type 2**: System has zero steady-state error to both a step input and a ramp input. It has a non-zero steady-state error to a parabolic input.

The following summary of the Steady-State Errors due to Step, Ramp and Parabolic inputs can be constructed, as shown in Table 5-1, where \( T \) denotes the type of system.

**Table 5-1: Steady-State Error analysis table.**

<table>
<thead>
<tr>
<th>Type</th>
<th>( K_p )</th>
<th>( K_v )</th>
<th>( K_a )</th>
<th>Step</th>
<th>Ramp</th>
<th>Parabolic (Acceleration)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T = 0 )</td>
<td>( K )</td>
<td>0</td>
<td>0</td>
<td>( e_{ss} = \frac{R}{1 + K_p} )</td>
<td>( e_{ss} = \infty )</td>
<td>( e_{ss} = \infty )</td>
</tr>
<tr>
<td>( T = 1 )</td>
<td>( \infty )</td>
<td>( K )</td>
<td>0</td>
<td>( e_{ss} = 0 )</td>
<td>( e_{ss} = \frac{R}{K_v} )</td>
<td>( e_{ss} = \infty )</td>
</tr>
<tr>
<td>( T = 2 )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( K )</td>
<td>( e_{ss} = 0 )</td>
<td>( e_{ss} = 0 )</td>
<td>( e_{ss} = \frac{R}{K_a} )</td>
</tr>
<tr>
<td>( T \geq 3 )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( e_{ss} = 0 )</td>
<td>( e_{ss} = 0 )</td>
<td>( e_{ss} = 0 )</td>
</tr>
</tbody>
</table>

**Example 5-2.** For the given system with its block diagram listed below:

![Block Diagram](image-url)
(A) Find the overall closed-loop transfer function.
(B) If \( T = 0 \), find the steady-state position error and velocity error.
(C) If \( T = 1 \), find the steady-state position error and velocity error.

Solution:

(A) One can obtain the overall transfer function by applying rules of simplifying block diagram.

\[
\begin{align*}
G(s) &= \frac{5}{s^2(1+s)+5s} \\
H(s) &= 1
\end{align*}
\]

One may first apply case 3 in Table 1-1 to obtain the transfer function for the inner closed-loop as

\[
G(s) = \frac{5}{s^2(1+s)+5s} = \frac{5}{s^2(1+s)+5s+5}.
\]

Then, use case 3 once again with a unity feedback (\( H(s)=1 \)), therefore, the overall transfer function can be obtained as

\[
\frac{C(s)}{R(s)} = \frac{5}{s^2(1+s)+5s+5} = \frac{5}{s^2(1+s)+5s+5}.
\]

Or, one can use Mason Rule and get the overall transfer function as

\[
\frac{C(s)}{R(s)} = \frac{5}{s^2(1+s)+5s+5} = \frac{5}{s^2(1+s)+5s+5}.
\]

(B) \( T = 0 \), the open-loop transfer function \( G(s)H(s) \) is

\[
G(s)H(s) = \frac{5}{(1+s)+5s} = \frac{5}{1+6s}.
\]

To find the steady-state position error, the input is assumed as a unit step input, \( R(s) = \frac{1}{s} \), and the position error constant can be calculated as
\[ K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \left[ \frac{5}{1 + 6s} \right] \cdot 1 = 5. \]

Therefore, the steady-state position error can be obtained, namely,
\[ e_{ss,p}(t) = \frac{1}{1 + K_p} = \frac{1}{1 + 5} = \frac{1}{6}. \]

To find the steady-state velocity error, the input is assumed as a unit ramp input,
\[ R(s) = \frac{1}{s^2}, \]
and the velocity error constant can be calculated as
\[ K_v = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} s \cdot \frac{5}{1 + 6s} = 0. \]

Therefore, the steady-state velocity error is
\[ e_{ss,v}(t) = \frac{1}{K_v} = \frac{1}{0} = \infty. \] 

(C) \[ T = 1, \] the open-loop transfer function \( G(s)H(s) \) is
\[ G(s)H(s) = \frac{5}{s(1 + s) + 5s} = \frac{5}{s(s + 6)}. \]

To find the steady-state position error, the input is assumed as a unit step input,
\[ R(s) = \frac{1}{s}, \]
and the position error constant can be calculated as
\[ K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \left[ \frac{5}{s(s + 6)} \right] \cdot 1 = \infty. \]

Therefore, the steady-state position error can be obtained, namely,
\[ e_{ss,p}(t) = \frac{1}{1 + K_p} = \frac{1}{1 + \infty} = 0. \]

To find the steady-state velocity error, the input is assumed as a unit ramp input,
\[ R(s) = \frac{1}{s^2}, \]
and the velocity error constant can be calculated as
\[ K_v = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} s \cdot \frac{5}{s(s + 6)} = \lim_{s \to 0} \frac{5}{s + 6} = \frac{5}{6}. \]

Therefore, the steady-state velocity error is
\[ e_{ss,v}(t) = \frac{1}{K_v} = \frac{1}{\frac{5}{6}} = \frac{6}{5}. \]
First-Order Systems:

The transfer function of a first-order system can be expressed as

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s\tau + 1},$$  \hspace{1cm} (5-22)

and, the output of the system can be written as

$$C(s) = \frac{1}{s\tau + 1} \cdot R(s).$$  \hspace{1cm} (5-23)

The unit step response of the system in time domain can be obtained by inverse of the Laplace transform, or using the formula listed in Table 4-1, namely,

$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s\tau + 1} \cdot \frac{1}{s}\right] = L^{-1}\left[\frac{1}{s} + \frac{-1}{s + \frac{1}{\tau}}\right] = 1 - e^{-\frac{t}{\tau}} \hspace{1cm} t \geq 0, \hspace{1cm} (5-24)$$

where $\tau$ is the time constant of the given control system. By plotting the output signal in time domain, Figure 5-5 can be obtained.

Classical Second-Order Systems:

Although true second-order control systems are rare in practice, their analysis generally helps to form a basis for understanding of design and analysis techniques. The transfer function of a classical second-order system can be represented as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$  \hspace{1cm} (5-25)
where

\[ \zeta > 1: \text{overdamped}, \]
\[ \zeta = 1: \text{critically damped}, \]
\[ \zeta < 1: \text{underdamped}, \]

and

\[ \omega_n = \text{undamped natural frequency}. \]

When \( \zeta < 1 \) and a unit step is injected into the system,

\[ c(t) = 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin\left(\sqrt{1 - \zeta^2} \omega_n t + \phi\right), \quad (5-26) \]

where

\[ \phi = \tan^{-1}\left(\frac{\sqrt{1 - \zeta^2}}{\zeta}\right). \quad (5-27) \]

The time domain plot for such a second-order system with a unit step input is shown in Figure 5-6.

![Figure 5-6. The typical unit step input response of a second-order system.](image)

There are some important performance criteria that are typically used to characterize the transient response to a unit step input include percent (maximum) overshoot, peak time, rise time and settling time.
Percent overshoot: the percentage of the maximum overshoot (the largest deviation of the output over the step input during the transient state).

\[
\%\text{OS} = e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}} \times 100\%. \quad (5-28)
\]

Peak time: The time it takes to reach the first peak.

\[
t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} \text{ seconds.} \quad (5-29)
\]

Rise time: The time required for the output to rise from 10% of the final value to 90% of the final values.

\[
t_r = \frac{0.8 + 2.5\zeta^-}{\omega_n} \text{ seconds.} \quad (5-30)
\]

Settling time: The time it takes for the system to settle within certain percentage of the final value.

\[
t_s = \frac{4}{\zeta \omega_n} \text{ seconds. (for 2%)} \quad (5-31)
\]

Example 5-3: The transfer function of the second order system given below is obtained in Example 1-1 as \( \frac{C(s)}{R(s)} = \frac{4}{s^2 + s + 4} \). What are the percent overshoot, peak time, rise time and the settling time for this system?

Solution:

By comparing the actual transfer function of the system with the classical second-order equation, namely,

\[
\frac{C(s)}{R(s)} = \frac{4}{s^2 + s + 4} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

One should be able to obtained the undamped natural frequency and the ramping ratio as \( \omega_n = 2 \); \( \zeta = \frac{1}{2\omega_n} = 0.25 \)
\[
\frac{C(s)}{R(s)} = \frac{4}{s^2 + s + 4} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

\[
\% \text{ Overshoot: } \% OS = e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{-\frac{0.25 \pi}{\sqrt{1-0.25^2}}} \times 100 = 44.43\% 
\]

\[
\text{Peak time: } t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{2\sqrt{1-0.25^2}} = 1.6223 \text{ seconds}
\]

\[
\text{Rise time: } t_r = \frac{0.8 + 2.5 \zeta}{\omega_n} = \frac{0.8 + 2.5 \times 0.25}{2} = 0.7125 \text{ second}
\]

\[
\text{Settling time: } t_s = \frac{4}{\zeta \omega_n} = \frac{4}{0.25 \times 2} = 8 \text{ seconds (for 2\%)}
\]

If one uses Matlab/Simulink to model this simple system, one should be able to get the plot shown above and confirm the answers obtained with equations (5-28)–(5-31).
Example 5-4: Find $K$ and $a$ for the closed-loop system below such that the transient response to a step input satisfies: $\%OS \leq 5\%$ & $t_s \leq 4$ seconds

![Control System Diagram](image)

Solution:

The transfer function of the system can be obtained as

$$\frac{C(s)}{R(s)} = \frac{K}{s(s + a)} \left( 1 + \frac{K}{s(s + a)} \right) = \frac{K}{s^2 + as + K} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Therefore,

$K = \omega_n^2$; $a = 2\zeta\omega_n$.

The percent overshoot is required to be less than or equal to 5%, hence,

$$\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% = 5\%$$

$$\Rightarrow \quad \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} = \ln(0.05) = -2.995732$$

$$\Rightarrow \quad (0.95357)\sqrt{1-\zeta^2} = \zeta$$

$$\Rightarrow \quad 0.9093 - 0.9093\zeta^2 = \zeta^2$$

$$\Rightarrow \quad \zeta^2 = \frac{0.9093}{1.9093}$$

$$\Rightarrow \quad \zeta = \sqrt{\frac{0.9093}{1.9093}} = \sqrt{0.47625} = 0.69$$

To find the undamped natural frequency, one needs to use the other given condition, namely,

$$t_s = \frac{4}{\zeta \omega_n} = 4$$

$$\Rightarrow \quad \zeta \omega_n = 1$$

$$\Rightarrow \quad \omega_n = \frac{1}{\zeta} = \frac{1}{0.69} = 1.4491$$
Now, one should be able to find K and \( a \) as follows:

\[
K = \omega_n^2 = 2.1
\]

and

\[
a = 2\zeta\omega_n = 2 \cdot 1 = 2
\]

Other than using Matlab/Simulink to verify the solution, one may use a spreadsheet to plot the output as given by equations (5-26) and (5-27).
References: